KAC'S FORMULA, VERTICES OF CHAINS, INFINITESIMAL MEASURES, AND EQUIDECOMPOSABILITY OF FUNCTIONS AND ENHANCED FUNCTIONS

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THIS WORK IS DEDICATED TO THE MEMORY OF MY DEAR FATHER, WHO DID NOT LIVE TO SEE IT COMPLETED.

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Synopsis

The present work hinges on the five parts of the title: Kac's formula, equidecomposability, vertices of chains, infinitesimal measures, enhanced functions.

The notion of chains and their vertices is proposed as a language to define and treat equidecomposability of functions on measure spaces acted measure-preservingly by groups. Thus, equidecomposable functions are vertices of the same chain. The fact that vertices of the same chain have the same integral is proposed as a way to get many formulas, including and generalizing Kac's formula, which states that when the acting group is $\mathbf{Z}$ and the measure is probability, the relative expectation of the return time to a measurable subset E with conull saturation is the reciprocal of the measure of E. This formula is viewed as a consequence of the fact that the return time (defined as zero outside E) and the characteristic function of the "set of points that visited E", differing from the saturation in a set of measure 0, are equidecomposable, a fact derived from an argument in the language of "chains" akin to the well-known Wright's proof using the "Kakutani skyscraper".

When G is discrete (which we shall refer to as the "discrete case"), the equality in integral of equidecomposable functions is trivial, so is the passage to the language of vertices of chains, yet this language is an easy path to find many formulas, developed in Section 1. These deal with $\mathbf{Z}$ as acting group, where one gets modifications of Kac's formula, as well as with multidimensional groups (speaking, for example, on "partially ordered Time"). One finds connections among attributes of (repeated) return and arrival in the $\mathbf{Z}$ case (cf. results of Kastelyn). One has here a kind of "toy" where possibly interesting formulas can emerge from the "playing".

It should be noted that ergodicity is usually irrelevant to the above considerations and assuming it seems of little help. Thus in Kac's Formula what matters is only that the saturation of E is conull.

What gives more emphasis to this language is the fact that it extends to the "continuous" case, to which Section 2 is devoted, and which for our purpose is Borel measure-preserving action of a 2nd-countable locally compact group G on a standard σ-finite measure space (one cannot go beyond locally compact groups, since Haar measure is crucial for our treatment). This allows us to formulate "continuous" Kac-like formulas in complete analogy with the discrete case (and with the prospect of getting more by "playing"), *provided G is unimodular*. But in order to do this and circumvent the obvious fact that simple-minded Kac formula fails for $\mathbf{R}$-actions (i.e. flows) – for $\mathbf{R}$ acting on the circle $\mathbf{T}$ by rotation and E composed of intervals the return time to E is $\neq 0$ only on a finite set – one uses the notions of "infinitesimal measures" and "enhanced functions" dealt with in Section 2. (These notions are akin to the concept of *Palm measure*, standard in the theory of point processes.) Thus, "infinite constants" which are mock Radon-Nikodym derivatives w.r.t. Haar measure of non-Radon invariant measures in G such as the counting measure, are uses to "enhance" functions with zero integral to "reveal" their "infinitesimal integral" – their integral w.r.t. an "infinitesimal measure" induced by the original (say, probability) measure. In the above example of rotation of the circle, the return time, having zero usual expectation, will nevertheless have "infinitesimal expectation" which relates to the measure of E by a Kac-like formula. Some examples of infinitesimal measures are given, some of them having differential-geometric aspects. As mentioned above, formulas analogous to the discrete case, but involving "enhanced functions" and "infinitesimal measures" are stated with proofs analogous to the proofs in the discrete case (modulo a "foundation" involving measure theory and descriptive set theory – see below). A kind of "multiple-dimensional continuous Kac arena" for G a unimodular Lie group is given by replacing the $\mathbf{Z}$- or $\mathbf{R}$- case "future until first return" by "the nearest point" w.r.t. a right-invariant Riemannian metric in G. This is applied in Section 4. (see below).

Working with chains in the "continuous case" consists of working with measures on G depending on parameters, in particular on the points of the standard Borel space acted by the group. These measures need not be σ-finite. Indeed, in most examples the most important deviation from the analogy with the discrete case is the participation of given non-σ-finite invariant measures on G, such as the counting measure. Thus the treatment splits into two parts of distinct flavour: the "formulas" part, with close analogy with the discrete case, the main difference being the appearance of "infinitesimal measures", and the "foundation" part, involving measure-theoretic and descriptive set-theoretic considerations, to which some paragraphs are devoted.

A feature of the language of vertices of chains is that its notions are independent of the measure (as long

as the action is measure-preserving, in other words, the measure is invariant), and are defined using just the action of the group on the Borel space (usually assume standard). For example, Kac's formula – the fact that the integral of the return time to a Borel set E is equal to the measure of the saturation of E, holds for any invariant measure, since the return time (defined as zero outside E) and the characteristic function of the set of points that visited E (a set differing from the saturation of E in a set null for every invariant probability measure) are equidecomposable.

Section 3 tries to formulate (in the discrete case) reverse implications: from equality of integral w.r.t. a comprehensive collection of invariant measures to being equidecomposable. We insist on the equidecomposability being via *non-negative* functions. Some theorems answering this are formulated and proved. In these theorems topological assumptions are made: it is assumed that the group acts continuously on a compact or locally compact space and the functions and chains are assumed upper or lower semi-continuous. (It is a well-known fact, proved by Varadarajan, that any standard Borel space acted in a Borel manner by a 2nd-countable locally compact group G can be embedded as an invariant Borel subset in a compact metric space on which G acts continuously.) Special attention is given to the special case of equidecomposability when one function is the finite average of translates of another. While most results are formulated for any acting group, in some matters *amenable* groups behave better. This connects with the Banach-Tarski paradox and Tarski's theorem, and also with "accumulating averages" ergodic theorems for general acting groups related to weak compactness and Ryll-Nardzewski's fixed-point theorem. The proofs in Section 3 use functional-analytic methods, mainly convex separation, sometimes taking the form of infinite dimensional minimax theorems analogous to Von Neumann's minimax theorem in Game Theory.

Section 4 deals with some aspects of a continuous counterpart of the subject of Section 3 – equidecomposable *enhanced functions* as independent of the original invariant measure. While many questions are raised, it is shown that the relation of equidecomposability is transitive, if some restrictions on the chains are imposed ("tame chains", which include most chains that we deal with in this work). It is also proved that in the case of unimodular *Lie groups* the original measure can be reconstructed from the infinitesimal measure and a related fact about finding functions enhanced by a given measure and equidecomposable with, say, a usual function. This is done using, for Lie groups G, sets with "discrete intersection with orbits" generalizing the Ambrose-Kakutani way of making a general flow a "flow under a function". To this known generalization of Ambrose-Kakutani (proved by Kechris, and by Feldman, Hahn and Moore) a proof is given (for Lie acting groups) using differential-geometric notions.

in Section 5 it is shown how our language may be used in proofs of results essentially the same as those of Helmberg and of Aaronson and Weiss, which relate to Kac's formula. Our treatment of the latter shows how the elementary (in the discrete case) method of vertices of chains can sometimes replace the pointwise ergodic theorem for multi-dimensional groups. This language is also used to get a proof of classical limit theorems of Renewal Theory[1], which proceeds completely analogously for discrete and continuous Time.

In the appendices some small theories needed or connected to the above are expounded.

[1]I am indebted to Prof. Jon Aaronson for his suggestion to apply the language of this work to Renewal Theory.

Notations

A group G (generally nonabelian) will be acting on a set Ω, always on the left, the action denoted by $(x,\omega)\mapsto x\omega = T^x\omega,\ x\in G, \omega\in\Omega$. Thus it is assumed that $e\omega=\omega$, $x(y\omega)=xy\omega$. A set endowed with such an action of G will be referred to as a G-set.

The unit element of G is denoted by e, but we shall also write 0 for the unit element (even in the non-abelian case) when it is considered as a "point" in G, not as an "acting agent".

Naturally, G acts (on the left) on functions f from Ω to, say, $\mathbf{R}$, by $xf(\omega)=f(x^{-1}\omega)$; Thus $\langle xf, x\omega\rangle = \langle x,\omega\rangle$. Further, G acts (on the left) on functionals on invariant spaces of such functions, e.g. on probability measures μ on a compact space on which G acts by homeomorphisms. Thus $\langle x\mu, f\rangle = \langle \mu, x^{-1}f\rangle$

The *orbit* of an $\omega\in\Omega$ is the set

$$\{x\omega : x\in G\}\subset\Omega$$

For $E\subset\Omega$, f a function on Ω and $\omega\in\Omega$, define:

$$\mathcal{O}_\omega E := \{x\in G : x\omega\in E\}\subset G$$
$$\mathcal{O}_\omega f := x\mapsto f(x\omega),\ \text{thus } \mathcal{O}_\omega f : G\to$$

Let $E\subset\Omega$. The **saturation** of E is defined by $\mathit{satur}\,E := \{\omega\in\Omega : \mathcal{O}_\omega E\neq\emptyset\} = \cup_{x\in G} xE$.

Often the group will act on a measure space $(\Omega,\mathcal{B},\mu)$, usually in a **measure-preserving** way, i.e. $\forall x\in G$ $\forall E\in\mathcal{B}$ also $xE\in\mathcal{B}$ and $\mu(xE)=\mu(E)$.

A **probability** (measure) space is a measure space with total mass 1. When dealing with such a space, probabilistic terminology will be used. For example, the **expectation** of a scalar- or vector-valued function (alias **stochastic variable** alias **random variable**) on Ω is just its integral.

A **null set** in a measure space Ω is a measurable subset $E\subset\Omega$ with measure 0. A **conull set** is a set whose complement is null.

A measure is **complete** if every subset of a null set is a measurable, hence a null set. The **completion** of a measure is the unique (complete) measure extending μ to the σ-algebra generated by the μ-measurable sets and the subsets of μ-null sets.

For a group G acting on a measure space $(\Omega,\mathcal{B},\mu)$, a measurable $E\subset\Omega$ is **almost-invariant** if $E\triangle xE$ is null for every $x\in G$. If G is countable, a set is almost-invariant iff it differs from an invariant set by a null set.

A **Borel structure** in a set Ω is a σ-algebra of subsets of Ω. A set with a Borel structure in it is called a **Borel space** and members of the σ-algebra are referred to as **Borel sets**. When a topological space is considered as a Borel space, it is understood, unless otherwise stated, that the Borel sets are the usual ones, i.e. the members of the σ-algebra generated by the open sets.

A mapping between two Borel spaces is **Borel** if the preimage of every Borel set is Borel.

Our Borel spaces will usually be **standard** – see §A.2.

A **standard measure space** is a standard Borel space with a measure on the Borel σ-algebra, sometimes the *completion* of such measure is understood. When the measure is probability, we speak of a **standard probability space**.

As usual, in a product of two standard Borel spaces Ω_1 and Ω_2 one takes as Borel structure the σ-algebra generated by the rectangles with Borel sides, i.e. the products $E_1\times E_2$, $E_1\subset\Omega_1$, $E_2\subset\Omega_2$ Borel sets.

An action of a group G with a standard Borel structure (say, a 2nd-countable locally compact group) on a standard Borel space Ω will be called **Borel action** if the mapping $(x,\omega)\mapsto x\omega : G\times\Omega\to\Omega$ is Borel. It will be said then that G acts **in a Borel manner** on Ω, and we shall speak of a **G-standard space** or a **standard G-space**.

Discrete subsets of topological spaces are assumed *closed*.

A **clopen** set is a simultaneously closed and open set.

A **meager** set in a topological space is a set contained in a countable union of closed sets with empty interior.

A **compact (locally compact) space** is a Hausdorff compact (locally compact) topological space. When a topological group (possibly discrete) G acts on it so that $(x,\omega)\mapsto x\omega : G\times\Omega\to\Omega$ is continuous, we speak of a **G-compact space (G-locally compact space)** or a **compact G-space**.

Similarly for other categories.

We adopt the Bourbaki notation for open and half-open intervals in a totally ordered set. Thus

$$]a,b[=\{x : a < x < b\} \qquad]a,b] = \{x : a < x \leq b\}$$

Some other notations will be

$\mathbf{Z}$	the set of integers
$\mathbf{Z}^+$	the set $\{0,1,2,\ldots\}$ of nonnegative integers
$\mathbf{N}$	the set $\{1,2,\ldots\}$ of positive integers
$\mathbf{R}$	the set of real numbers
$\mathbf{R}^+$	$=\{x \in \mathbf{R} : x \geq 0\}$
$\overline{\mathbf{R}^+}$	$=\mathbf{R}^+ \cup \{+\infty\}$
$\mathbf{T}$	the circle group $\mathbf{R}/\mathbf{Z}$
$\# E$	$(\in \mathbf{Z}^+ \cup \{\infty\})$ the number of elements of a set E
E^c	the complement of the set E
$\triangle$	symmetric difference of sets
1_E	the characteristic function of the set E
$\overline{E}$	is the closure of the set E
E°	is the interior of the set E
∂E	is the boundary of the set E
δ_x	the Dirac measure at x
$\text{count} = \text{count}_X$	the counting measure on the set X
$\Pr$	the probability of an event
$\mathbf{E}$	the expectation of a stochastic variable
$*$	convolution
$\wedge$	logical "and"; exterior multiplication; lattice operation $\min$
$\vee$	logical "or"; lattice operation $\max$
&	logical "and"
V^*	the dual of a (normed) vector space V
$\langle , \rangle$	the result of an element of a dual space applied to an element of a space
$\mathcal{C}(\Omega)$	the space of continuous functions on a compact Ω
$\mathcal{C}^+(\Omega)$	the cone of nonnegative functions in $\mathcal{C}(\Omega)$
$\mathcal{C}_{00} = \mathcal{C}_{00}(\Omega)$	the space of continuous functions with compact support on a locally compact Ω
$\mathcal{C}_{00}^+$	the cone of nonnegative functions in $\mathcal{C}_{00}$
$\text{Ł}^1(\Omega)$	the space of integrable functions on a measure space Ω
$\text{Ł}^\infty(\Omega)$	the space of bounded measurable functions on a measure space Ω
$\hat{G}$	the dual group of a locally compact abelian group G
$\hat{f}$	the Fourier transform of the function f
Δ	the modular function of a locally compact group

A notational convention, explained in the text, is:

When (see §2.4) λ is a left Haar measure on the acting group G and Λ some other (not necessarily σ-finite) Δ-right invariant measure, $f\frac{d\Lambda}{d\lambda}$ denotes the function f "enhanced" by $\frac{d\Lambda}{d\lambda}$; $\frac{d\Lambda}{d\lambda}\mu$ denotes the "infinitesimal measure" equal to the measure μ in Ω enhanced by $\frac{d\Lambda}{d\lambda}$

We use the following abbreviations: **resp.** – *respectively*; **s.t.** – *such that* or *so that*; **w.r.t.** – *with respect to*; **w.l.o.g.** – *without loss of generality*; **t.f.a.e.** – *the following are equivalent*; **a.e.** – *almost everywhere*; **a.a.** – *almost all*; **a.s.** – *almost surely*; **p.m.** – *probability measure*; **i.i.d.** – *independent identically distributed*; **u.s.c.** – *upper semi-continuous*; **(b.)l.s.c.** – *(baire) lower semi-continuous.*

1 The Discrete Case: Chains, Hypergraphs, Expectation of Vertices

1.1 Equidecomposability and Chains, The VE and HG Theorems

The impetus for this work has been an attempt to generalize Kac's formula [Kac] to more general acting groups, in particular to "multi dimensional" groups and "continuous" groups.

Kac's formula can be stated as follows: for a measure-preserving invertible transformation T acting on a probability space $(\Omega, \mathcal{B}, \mu)$, and for a measurable $E \subset \Omega$, the following two functions have the same integral:

- $\rho_E(\omega)$, **the return time to** E, defined for $\omega \in E$ as the first $k > 0$ s.t. $T^k\omega \in E$, as ∞ if $\omega \in E$ and $T^k\omega \notin E, k = 1, 2, \ldots$ and as 0 outside E;
- The characteristic function $1_{satur\,E}$ of the smallest (measurable) invariant set containing E (note that for ergodic action and $\mu(E) > 0$ $satur\,E$ is conull. Yet $satur\,E$ may be conull without ergodicity, e.g. take $\Omega = \mathbf{T} \times \mathbf{T} =$ the torus, $\mathbf{Z}$ acting by irrational rotation *of the first coordinate* and $E = I \times \mathbf{T}$, I an interval).

Note that the sets

$$\{\omega : \{k : T^k\omega \in E\} \text{ has a biggest (resp. smallest) element } n\}$$

form an infinite sequence of disjoint sets with the same measure, hence are all null (this is Poincaré's Recurrence Theorem). Therefore ρ_E is finite a.e. This consideration also allows us to replace $satur\,E$ by $satur'E$, to be defined as the set of all ω that has visited E in the past or present, i.e.

$$satur'E := \{\omega \in \Omega : \mathcal{O}_\omega E \cap\,]-\infty, 0] \neq \emptyset\} \tag{1}$$

It is helpful to view Kac's formula as a consequence of ρ_E and $1_{satur'E}$ being, as we shall show, (infinitely, via measurable functions) **equidecomposable** i.e. there exists a family $(f_k)_{k\in\mathbf{Z}}$ of nonnegative measurable functions s.t.

$$\forall \omega \in \Omega \quad \rho_E(\omega) = \sum_{k\in\mathbf{Z}} f_k(\omega), \quad 1_{satur'E}(\omega) = \sum_{k\in\mathbf{Z}} f_k(T^{-k}\omega) \tag{2}$$

The way we choose to demonstrate this equidecomposability, which is sometimes the best way to describe equidecomposability, is as follows:

Expand any "sequence" f_k to a "matrix"

$$F_{k,l}(\omega) = f_{l-k}(T^k\omega) \tag{3}$$

This matrix has the invariance property:

$$F_{k+n,l+n}(\omega) = F_{k,l}(T^n\omega) \tag{4}$$

and one easily sees that every "matrix" satisfying (4) comes from some sequence $f_k = F_{0,k}$.

Call an ordered pair (k, l) of elements of the group $\mathbf{Z}$ a 1-*simplex*, and consider the *invariant (in the above sense)* 1-*chain* $\sum_{k,l} F_{k,l} \cdot (k, l)$. This may be thought of as a chain, (i.e. formal sum of simplices) with functions on Ω as coefficients (with the group $\mathbf{Z}$ acting on these functions), or as a function from Ω to the space of chains with scalar coefficients. This is in accordance with one of the ways to treat group cohomology (compare the treatment in [Wl], Ch. IX §3). *Invariant m-chains* ($m = 0, 1, 2, \ldots$) are defined analogously, as "sums" of m-simplices $(k_0, k_1, \ldots, k_n)$, $k_i \in \mathbf{Z}$ with $m+1$-dimensional "matrices" $F_{k_0,k_1,\ldots,k_m}(\omega)$ of coefficients, satisfying the invariance property

$$F_{k_0+n,\ldots,k_m+n}(\omega) = F_{k_0,\ldots,k_n}(T^n\omega) \tag{5}$$

or in words:

$$\begin{array}{l}\text{for a simplex in the chain of } T^n\omega,\\ \text{we have the } n\text{-shifted simplex in the chain of } \omega.\end{array} \tag{6}$$

In particular, a single function g is "equivalent" to an invariant 0-chain $\sum_k G_k(\omega) \cdot (k) = \sum_k g(T^k\omega) \cdot (k)$.

Now, one easily checks that the above-mentioned property of functions $g(\omega) = \sum_k f_k(\omega)$ and $g'(\omega) = \sum_k f_k(T^{-k}\omega)$ to be equidecomposable via the sequence $f_k(\omega)$ can be expressed using the 1-chain F and the 0-chains G and G' corresponding to f, g and g' resp. Namely, the coefficients in G' are the *row-sums* and those in G the *column-sums* of the coefficient matrix in F. In other words, G' and G are the two *vertices* - the *source* and the *target* - of F. To define the vertices of a 1-chain, note that every 1-simplex (k,l) has two vertices (these being, of course, 0-simplices): the source (k) and the target (l), and this is extended to chains by linearity and summability. (By invariance, to check that an invariant 0-chain is a vertex of an invariant 1-chain it is enough to check one coefficient of the 0-chain, say, the coefficient at (0). This coefficient is just the function corresponding to the 0-chain.)

One should be warned that this use of the word "vertices" is at variance with the common use in Graph Theory, where the vertices of a graph are all the 0-simplices occuring in the graph (thus in the above chains these "vertices" are all the elements of the acting group $\mathbf{Z}$). Still, we stick to our use of this term and it seems not to cause misunderstanding.

Now, the definition of invariant m-chain carries over to **general discrete groups** - not necessarily abelian, where if the group acts on the left *right shifts* are required in (6), i.e. one requires

$$F_{x_0 y,\dots,x_m y}(\omega) = F_{x_0,\dots,x_m}(y\omega) \quad x_0,\dots,x_m, y \in G \; \omega \in \Omega \tag{7}$$

Treating equidecomposable functions as vertices of an invariant chain, the (trivial) fact that they have the same integral (= expectation) is encoded in the following formulation, which, as we shall see, is amenable to various applications in the discrete case and can be carried over to the "continuous" case:

The Vertices Expectation (VE) Theorem (Discrete Case) *Suppose a countable discrete group G acts measure-preservingly on a probability space Ω, and a nonnegative (right-)invariant m-chain depending on $\omega \in \Omega$ measurably is given. (right-)invariance means that (7) is satisfied. Then all the vertices of the m-chain have the same expectation.*

Proof an "invariant" m-chain is invariant only "globally" - the chain of a single ω is not invariant (shifting the chain by $x^{-1} \in G$ replaces ω by $T^x\omega$) but its expectation (which is an m-chain, of course not dependent on ω), enjoys "genuine" invariance - for every simplex in the chain we have all its right-shifts in the chain with the same coefficient. Therefore it is a countably infinite linear combination of "right-diagonal chains", (these being the sums of all the right-shifts of one simplex). Since all the vertices of a right-diagonal chain are the same (being equal to the 0-chain $\sum_{x\in G}(x)$) we are done.

QED

Note that in the VE thm. the assumption that Ω is probability is superfluous. Ω can have σ-finite measure. Only then one has to speak of "integral" instead of "expectation".

Let us return to the Kac case:

The 1-chain F we take here is, for each ω,

$$\begin{array}{l} \text{the sum (=collection) of the 1-simplices (=arrows) } (k,l) \\ \text{emanating from some } k \in \mathbf{Z} \\ \text{to the last } l \in \mathbf{Z} \text{ s.t. } l \le k \text{ and } T^l\omega \in E. \end{array} \tag{8}$$

The invariance (6), i.e. (4) is clear. Also, it is evident that the source of F corresponds to $1_{satur'E}(\omega)$ and the target to $\rho_E(\omega)$ (check the coefficient at (0) ! recall that the coefficient at (0) of the source (target) is the # of arrows with source (target) (0)). Consequently, by VE, we have Kac's formula.

This proof is very close to Wright's proof which uses the Kakutani skyscraper (see [Pe], pp. 45–46).) Indeed, the arrows of our graph "do the construction work" of the Kakutani skyscraper by "pushing" to $0 \in \mathbf{Z}$ all the $k \in \mathbf{Z}$ s.t. 0 is the last $l \in \mathbf{Z}$ s.t. $l \le k$ and $T^l\omega \in E$, thus "piling" all of Ω over E.

In the Kac case, the chain was of a special kind: for every ω we had a set of simplices, and the chain was their sum, i.e. the coefficient matrix was the characteristic function of this set. A set of m-simplices in G (possibly depending on ω) is called a (directed) m**-hypergraph** on G (for $m = 1$ – a (directed) **graph**), and (in the discrete case) *every hypergraph is (identified with) a chain.*

So, as a particular case of the VE thm., which has Kac's as a special case, we have

The Hypergraph (HG) Theorem *Let a countable discrete group G act measure-preservingly on a probability space Ω.*

Suppose we are given an m-hypergraph on G, depending on $\omega \in \Omega$ measurably and (right-)invariantly. (invariance means: the hypergraph for ω is the right x-shift of the hypergraph for $x\omega$, $\forall x \in G, \omega \in \Omega$).

Then for $i = 0, 1, \ldots, m$, the expectation of the number of simplices having (0) as their i-th vertex is the same for all i's.

Again, the HG thm. holds when Ω has σ-finite measure, speaking of "integral" instead of "expectation".

Remark 1.1.1 In some particular cases (for general G), one refers, as in the Kac case, to a measurable $E \subset \Omega$, and the graph (for each fixed ω) has the property that every arrow ends in an element of $\mathcal{O}_\omega E$ and every $x \in \mathcal{O}_\omega E$ is the end of a *unique* arrow. Then one has a "Kac formula": the expectation of the number of arrows ending in 0 (that being a function of ω) is $\mu(satur\, E)$. One such case is the treatment of Aaronson and Weiss' "Kac functions" in §5.2.

Remark 1.1.2 The notion of an invariant m-chain is adapted to viewing a G-set as a *groupoid*.

A set Ω acted by a group G may be treated as a *groupoid* Γ with base Ω (see [Wn]), A groupoid Γ with base Ω being just a small category with set of objects Ω in which every morphism is invertible, Γ being the set of morphisms. In the case of a G-set Ω, Γ is the set of "arrows" (ω_1, x, ω_2), $\omega_1, \omega_2 \in \Omega$, $x \in G$, $\omega_2 = x\omega_1$. In this way the "ω-independent" character of an $x \in G$ as an "acting agent" is lost, thus if G acts freely the groupoid structure encodes just the orbit equivalence relation and is isomorphic to a *subgroupoid* of the groupoid $\Omega \times \Omega$, where a unique morphism for every $\omega_1 \to \omega_2$ is understood (the subgroupoids of $\Omega \times \Omega$ are just the equivalence relations in Ω).[2] On the other hand, the groupoid structure is more flexible: for instance, for $E \subset \Omega$ we always have the **induced groupoid** Γ_E, which in case of a $\mathbf{Z}$-action will correspond to the *induced transformation* (see [Pe] p.12).

Thus, instead of m-simplices in G one may consider m-simplices in Γ, defined as morphisms from the groupoid $\{0, 1, \ldots, m\} \times \{0, 1, \ldots, m\}$ to Γ. Such an m-simplex corresponds to an orbit of G acting on pairs consisting of an $\omega \in \Omega$ and an m-simplex in G. Hence, invariant ω-dependent m-chains on G, as defined above, are in one-one correspondence with m-chains on Γ. The *vertices* of such chains turn up to be functions on Ω, corresponding to the vertices of invariant ω-dependent m-chains on G.

This gives another merit to the treatment of equidecomposability via 1-chains: it carries over straightly to groupoids.

In fact, since equidecomposable functions can be defined in a groupoid, although translates of a function have no meaning, the holding of VE for $m = 1$ becomes the *definition* of measure-preservingness of the groupoid.

1.2 An Assortment of Kac-like Theorems, Discrete Case

In this §, the setting is a that of group G acting measure-preservingly on a probability space $(\Omega, \mathcal{B}, \mu)$.

1.2.1 One-Dimensional (Discrete) Examples

The VE Thm., in particular the HG Thm., specialize, for $\mathbf{Z}$ as well as for "multi-dimensional" groups, to many cases which are of interest. Let us list some of them.

Recall that the source of an invariant 1-chain F is an invariant 0-chain, determined by its coefficient at (0) which is the sum of coefficients of all edges (arrows) in F with source (0); similarly for targets of 1-chains and vertices of m-chains.

1. $G = \mathbf{Z}$; $E \in \mathcal{B}$; given a measurable nonnegative function f on E.

   the 1-chain F = the sum of arrows (k, l) s.t. $T^k\omega \in E$ and l is the first $l < k$ with $T^l\omega \in E$, with coefficient $f(T^k\omega)$.

[2] Investigations initiated in [FM1] and [FM2] have shown that for many purposes it suffices, instead of the structure of a G-Borel space (G countable), to consider just the groupoid, i.e., in case of a free action, the orbit equivalence relation.

Then: If $\omega \in E$, the (coefficient of) the source at (0) is $f(\omega)$, while the coefficient of target at (0) is $f(T_E(\omega))$ where T_E is the **induced transformation** (see [Pe] p.12). For $\omega \notin E$ both the source and the target at (0) are 0.

By VE $\int_E f(\omega) = \int_E f(T_E(\omega))$, so we have proved that *the induced transformation is measure-preserving.*

2. $G = \mathbf{Z}$; $E \in \mathcal{B}$. Let $n \in \mathbf{Z}^+$. As a variation on (8), take the graph:

$$F = \{(k,l) : k-l = n \,, T^l\omega \in E \,, \forall j \in]l,k]\; T^j\omega \notin E\}$$

Then: there is one arrow with *target* (0) if $\omega \in E$ and $n < \rho_E(\omega)$, otherwise there is no such arrow.

Define the **arrival time** $\xi_E(\omega)$ of an $\omega \in \Omega$ as the minimal $k = 0, 1, \dots$ s.t. $T^k\omega \in E$ (and as $+\infty$ if $\exists$ no such k). In particular $\xi_E(\omega) = 0 \Leftrightarrow \omega \in E$. ξ_E is finite a.e. on *satur* E and $+\infty$ outside *satur* E.

Then there is one arrow with *source* (0) if $\xi_E^{(-)}(\omega) = n$, otherwise none, when **a superscript** $(-)$ denotes entities referring to **the inverse action** T^{-1}.

so we conclude the following strengthening of Kac's:

$$\begin{array}{l}\text{The probability that } \rho_E > n \ (n = 0, 1, \dots) \\ \text{is equal to the probability that } \xi_E^{(-)} = n.\end{array} \tag{9}$$

So, knowing the distribution function of one of ρ_E, $\xi_E^{(-)}$ gives us that of the other one.

3. Invoking the inverse action in (9) can be avoided:

Consider the following two graphs:

$$F' = \{(k,l) \in \mathbf{Z}^2 : k-l = n,\ T^k\omega, T^l\omega \in E, \forall j \in]k,l[\; T^j\omega \notin E\} \quad n = 1, 2, \dots$$
$$F'' = \{(k,l) \in \mathbf{Z}^2 : k-l = n,\ \forall j \in [k,l]\; T^j\omega \notin E\} \quad n = 0, 1, 2, \dots$$

The target and source of F' at (0) give the probabilities for $\rho_E = n$ and for $\rho_E^{(-)} = n$, resp., while the target and source of F'' at (0) give the probabilities for $\xi_E > n$ and for $\xi_E^{(-)} > n$, resp. By HG, we have:

Proposition 1.2.1 *ρ_E and $\rho_E^{(-)}$ have the same distribution, similarly ξ_E and $\xi_E^{(-)}$ have the same distribution.*

Hence we may replace (9) by the following strengthening of Kac's:

Proposition 1.2.2 *The probability that $\rho_E > n$ $(n = 0, 1, \dots)$ is equal to the probability that $\xi_E = n$.*

4. To obtain another formulation of (9), let $s : \mathbf{Z}^+ \to \overline{\mathbf{R}^+}$ be a nonnegative sequence, let $S(n) = \sum_{0 \le k < n} s(k)$.

Take the chain with same arrows as in (8), i.e. the (k,l) with l the last $l \le k$ s.t. $T^l\omega \in E$, but with coefficients $s(k-l)$.

The source at $(0) = s(\xi_E^{(-)}(\omega))$; the target at $(0) = S(\rho_E(\omega))$.

Using VE and arguing as in 3, one obtains:

Proposition 1.2.3 *(another formulation of Prop. 1.2.2 and 1.2.1) Let $s : \mathbf{Z}^+ \to \overline{\mathbf{R}^+}$ be a nonnegative sequence. Let $S(n) = \sum_{0 \le k < n} s(k)$. Then $s(\xi_E(\omega))$, $s(\xi_E^{(-)}(\omega))$ $S(\rho_E(\omega))$ and $S(\rho_E^{(-)}(\omega))$ have the same expectation.*

Kac's is the case $s \equiv 1$.

Corollary 1.2.4 *If satur E is conull, then for any $p \geq 0$, $\rho_E \in L^{p+1}(\Omega) \Leftrightarrow \xi_E \in L^p(\Omega)$.*

Remark 1.2.5 Since ξ_E is obviously in $L^0(\Omega)$, we have ρ_E is in $L^1(E)$ which is included, of course, in Kac's. One cannot say anything further, because for any $g : [0,1] \to \mathbf{Z}^+$ with integral 1 one can construct Ω, E with *satur* $E = \Omega$ and ρ_E having the same distribution as g. This is done using the discrete "flow under a function" construction (see [Pe], p. 11). So ξ_E need not be integrable.

In case ξ_E is integrable, equivalently, by the above Corollary, ρ_E is in L^2, one can give the following proof to Kac's formula :

Proof Assuming E conull and ξ_E integrable, Kac's formula follows from the equality:

$$\rho_E(\omega) = \xi_E(T\omega) + 1 - \xi_E(\omega)$$

QED

5. A combination of items 4 and 1:

   Let $E \in \mathcal{B}$ with *satur* E conull, $f : \Omega \to \mathbf{R}^+$ measurable and $s : \mathbf{Z}^+ \to \overline{\mathbf{R}^+}$, as in item 4.

   The chain: our "Kac" set of simplices, namely the (k,l) with l the last $l \leq k$ s.t. $T^l\omega \in E$, but with coefficients $f(T^k\omega)s(k-l)$.

   The source at (0): $f(\omega)s(\xi^{(-)}\omega)$.

   The target at (0): for $\omega \in E$, $\sum_{0 \leq k < \rho(\omega)} f(T^k\omega)s(k)$; 0 outside E.

   And one obtains:
   $$\int_\Omega f(\omega)s(\xi^{(-)}\omega) = \int_E \sum_{0 \leq k < \rho(\omega)} f(T^k\omega)s(k) \tag{10}$$

6. Another combination of items 4 and 1:

   Let $E \in \mathcal{B}$ with *satur* E conull, $f : E \to \mathbf{R}^+$ measurable and let $s : \mathbf{Z}^+ \to \overline{\mathbf{R}^+}$ and $S(n) = \sum_{0 \leq k < n} s(k)$ be as in item 4.

   The chain: similarly to item 5 – the "Kac" set of simplices: the (k,l) with l the last $l \leq k$ s.t. $T^l\omega \in E$, with coefficients $f(T^l\omega)s(k-l)$.

   The source at (0): $f\left(T^{-\xi^{(-)}\omega}(\omega)\right) s(\xi^{(-)}\omega)$.

   The target at (0): for $\omega \in E$, $f(\omega)S(\rho(\omega))$; 0 outside E.

   And one obtains:
   $$\int_\Omega f\left(T^{-\xi^{(-)}\omega}(\omega)\right) s(\xi^{(-)}\omega) = \int_E f(\omega)S(\rho(\omega)) \tag{11}$$

   Remark 1.2.6 For $\mathbf{Z}$-action and *satur* E conull, the system (ω, μ, T) can be recovered from the system $(E, \mu(\ |E), T_E)$ (the measure is conditional probability w.r.t. E) and the function $\rho_E : E \to \mathbf{N}$, via the discrete analog of the "flow under a function" construction (see [Pe], p. 11). Note that by Kac's, ρ_E has integral $1/\mu(E)$ on E, so we have a kind of reciprocity between the integral of the given function on one side and the measure of the given set on the other side.

7. **Proposition 1.2.7** – **return and arrival to two sets** *Let $\mathbf{Z}$ act in a measure-preserving manner on (Ω, μ). Let $E_1 \subset \Omega$ and $E_2 \subset \Omega$ be measurable.*

   Then:
   $$\begin{aligned} &\textstyle\int_{E_2 \cap \{0 < \xi_{E_1} \leq \rho_{E_2}\}} [\xi_{E_1} - 1] = \int_{E_1 \cap \{0 < \xi^{(-)}_{E_2} \leq \rho^{(-)}_{E_1}\}} \left[\xi^{(-)}_{E_2} - 1\right] = \\ &= \mu\left[satur E_1 \cap satur E_2 \cap \{0 < \xi_{E_1} \leq \xi_{E_2}\} \cap \{0 < \xi^{(-)}_{E_2} \leq \xi^{(-)}_{E_1}\}\right] \end{aligned} \tag{12}$$

Proof Take the (ω-dependent) 2-hypergraph consisting of the 2-simplices $(k_0, k_1, k_2) \in \mathbf{Z}^3$ s.t. $k_2 < k_0 < k_1$, $[k_2, k_1[\cap \mathcal{O}_\omega E_2 = \{k_2\}$ and $]k_2, k_1] \cap \mathcal{O}_\omega E_1 = \{k_1\}$. The three expressions in (12) are the expectations of its 2-, 1- and 0-th vertices resp.

QED

One can try to formulate other "variations" on this theme.

1.2.2 The Common Distribution of Repeated Return and Arrival

We remain in the case of $\mathbf{Z}$-action on a probability space (Ω, μ).

Let $E \subset \Omega$ be measurable. Let T_E be the induced transformation (see §1.2.1). For $\omega \in E$, we have its *return time* $\rho^{(1)}(\omega) = \rho_E(\omega)$, its *2nd return time* $\rho^{(2)}(\omega) = \rho_E(T_E\omega)$, its *3rd return time* $\rho^{(3)}(\omega) = \rho_E(T_E^2\omega)$ etc. The same for the inverse transformation $\rho^{(-)(1)(\omega)} = \rho_E^{(-)}(\omega)$, $\rho^{(-)(2)}(\omega) = \rho_E^{(-)}(T_E^{-1}\omega)$ etc.

For any $\omega \in \Omega$, we have its *arrival time* $\xi^{(1)}(\omega) = \xi_E(\omega)$, then the *subsequent return time* $\xi^{(2)}(\omega) = \rho_E(T^{\xi_E(\omega)}\omega)$, the *2nd subsequent return time* etc. The same for the inverse transformation.

What can be said about the joint distribution of these four sequences of stochastic variables? This is the theme of the following theorem, which is a direct application of HG (except item f.) ([Kas] gives formulas describing various aspects of these distributions. The repeated return times are discussed in [Br], Ch. 6).

Theorem 1.2.8 *Retaining the above notations,*

a. *Let $m > 0$, $r_1, \dots, r_m > 0$, $0 \le p \le m$, be integers.*

$$\begin{aligned}&\mu\{\omega \in E : \rho^{(1)} = r_1, \rho^{(2)} = r_2, \dots, \rho^{(m)} = r_m\} = \\ &= \mu\{\omega \in E : \rho^{(-)(1)} = r_m, \rho^{(-)(2)} = r_{m-1}, \dots, \rho^{(-)(m)} = r_1\} = \\ &= \mu\{\omega \in E : \rho^{(1)} = r_{p+1}, \rho^{(2)} = r_{p+2}, \dots, \rho^{(m-p)} = r_m, \rho^{(-)(1)} = r_p, \rho^{(-)(2)} = r_{p-1}, \dots, \rho^{(-)(p)} = r_1\}\end{aligned}$$

Denote this value by $P[r_1, r_2, \dots, r_m]$.

b. *Let $m > 0$, $r_1 \ge 0, r_2 \dots, r_m > 0$, be integers.*

$$\mu\{\omega \in E : \rho^{(1)} > r_1, \rho^{(2)} = r_2, \dots, \rho^{(m)} = r_m\} = \mu\{\omega \in \Omega : \xi^{(1)} = r_1, \xi^{(2)} = r_2, \dots, \xi^{(m)} = r_m\}$$

c. *Let $m, m' > 0$, $r_1 \ge 0, r_2 \dots, r_m > 0$, $r'_1 \ge 0, r'_2, \dots, r'_{m'} > 0$, be integers.*

$$\begin{aligned}&\mu\{\omega \in \Omega : \xi^{(1)} = r_1, \rho^{(2)} = r_2, \dots, \xi^{(m)} = r_m, \xi^{(-)(1)} = r'_1, \xi^{(-)(2)} = r'_2, \dots, \xi^{(-)(m')} = r'_{m'}\} = \\ &= P[r'_{m'}, \dots, r'_2, r_1 + r'_1, r_2, \dots, r_m]\end{aligned}$$

(note that this depends only on the sum *of the values r_1 and r'_1 for the arrival and inverse arrival times.)*

d. *Each one of the joint distribution of the $\rho^{(j)}$ or the joint distribution of the $\xi^{(j)}$ determines the joint distribution of all the $\rho^{(j)}$'s, $\xi^{(j)}$'s, $\rho^{(-)(j)}$'s and $\xi^{(-)(j)}$'s.*

e. *Assume $\mu(E) > 0$, $\mu(satur\, E) = 1$.*

   The joint distribution of $\rho^{(1)}, \rho^{(2)}, \dots$, w.r.t. the conditional probability in E, is stationary (that is, the image measure that they define in $\mathbf{N}^{\mathbf{N}}$ is invariant w.r.t. the shift $(\alpha_j) \mapsto (\alpha_{j+1})$). Note that by Kac', each $\rho^{(j)}$ has expectation $1/\mu(E)$.

f. *([Br] §6.10) The statement of e. is the only restriction on the joint distribution of the $\rho^{(j)}$, namely, any shift-invariant measure on $\mathbf{N}^{\mathbf{N}}$ with integrable coordinates can be realized as the image of $\rho^{(j)}$'s as in e.*

Proof Formulas a.-c. follow from HG:

For a., the hypergraph (depending on ω) consists of all m-simplices $(k_0, k_1, \ldots, k_m)$, $k_0 < k_1 < \ldots < k_m$ with $k_1 - k_0 = r_1, k_2 - k_1 = r_2, \ldots, k_m - k_{m-1} = r_m$, $T^{k_j}\omega \in E, 0 \le j \le m$, and $T^j\omega \notin E$ for j between any two adjacent k_j's. Now apply HG.

The hypergraph for b. is defined exactly the same, only here it is assumed $T^{k_i}\omega \in E, 0 < i \le m$ but $T^{k_0}\omega \notin E$. HG gives:

$$\mu\{\omega \in E : \rho^{(-)(1)} > r_1, \rho^{(1)} = r_2, \ldots, \rho^{(m-1)} = r_m\} = \mu\{\omega \in \Omega : \xi^{(1)} = r_1, \rho^{(2)} = r_2, \ldots, \rho^{(m)} = r_m\}$$

and to get b. use a.

For c., the hypergraph consists of the $(m+m'+1)$-simplices $(k_{-m'}, \ldots, k_{-1}, k_0, k_1, \ldots, k_m)$, with $k_j - k_{j-1} = r_j, k_{-j+1} - k_{-j} = r'_j, j > 0$, $T^j\omega \notin E$ for j between any two adjacent k_j's, $T^{k_j}\omega \in E, j \ne 0$ but $T^{k_0}\omega \notin E$.

d. follows from a.-c.

To prove e., take $p = 1$ in a. and sum over $r_1 \in \mathbf{N}$, to obtain:

$$\begin{aligned}&\tfrac{1}{\mu(E)}\mu\{\omega \in E : \rho^{(1)} = r_1, \rho^{(2)} = r_2, \ldots, \rho^{(m)} = r_m\} = \\ &= \tfrac{1}{\mu(E)}\mu\{\omega \in E : \rho^{(1)} = r_2, \rho^{(2)} = r_3, \ldots, \rho^{(m-1)} = r_m\}\end{aligned}$$

To prove f. proceed by the method of [Ne]:

We are given a probability measure on $\mathbf{N}^{\mathbf{N}}$, and a measure-preserving action of $\mathbf{Z}^+$ on $\mathbf{N}^{\mathbf{N}}$ by the shift $(T(\alpha))_j := \alpha_{j+1}$, $\alpha \in \mathbf{N}^{\mathbf{N}}$. First replace it by the unique shift-invariant measure on $\mathbf{N}^{\mathbf{Z}}$ whose image by the projection $\pi : \mathbf{N}^{\mathbf{Z}} \to \mathbf{N}^{\mathbf{N}}$ is the given measure. (To get the mass of a cylinder in $\mathbf{N}^{\mathbf{Z}}$, i.e. a set depending on a finite number of coordinates, write it as $T^{-k}\pi^{-1}C$ for a cylinder C in $\mathbf{N}^{\mathbf{N}}$.) Now, following [Ne], note that on the space X of all strictly increasing sequences of intgers $(x_n, n \in \mathbf{Z})$ $\mathbf{Z}$ acts in two commuting ways: first by the shift $(T(x))_j := x_{j+1}$ and second by the translation $(S(x))_j := x_j + 1$. Each action has a Borel section, hence a standard Borel space of orbits: for S the space of orbits is identified with $\mathbf{N}^{\mathbf{Z}}$ via $x \mapsto \alpha$ where $\alpha_j := x_{j+1} - x_j$, and the identification commutes with the shifts, while for T the space of orbits is identified with the space Ω of all subsets of $\mathbf{Z}$ which are unbounded above and below, this identification transferring S into the shift in Ω $\sigma \mapsto \sigma + 1$, $\sigma \in \Omega$. Now the given probability measure on $\mathbf{N}^{\mathbf{Z}}$, viewed as the space of orbits of X w.r.t. S, induces a probability measure on X (the integral of a measurable function on X being equal to the integral of its sums on orbits), this measure is shift-invariant on X, and similarly this measure in its turn induces a probability measure on the space Ω of T, the latter being shift-invariant. For this space Ω, with $T =$ the shift and $E := \{\sigma \in \Omega, 0 \in \sigma\}$, the $\rho_E^{(j)}$ will be distributed as the α_j.

QED

Remark 1.2.9 Take $m = m' = 1$ in c. (Thm. 1.2.8). This gives for $k > 0$ and any $r, r' \ge 0, r + r' = k$ that

$$\mu\{\omega \in \Omega : \xi = r, \xi^{(-)} = r'\} = \mu\{\omega \in E : \rho = k\} \tag{13}$$

Summing over $r > 0, r' > 0$ one gets $\int_E(\rho - 1) = \mu((satur\, E) \setminus E)$, i.e. Kac's. (13) is a kind of "decomposition" of Kac's.

1.2.3 "Multidimensional" Preordered Groups ("Relativistic" Time)

Natural generalizations of Kac's formula and its "refinements" such as Prop. 1.2.2 arise by letting "past" and "future" refer to a shift-invariant preordering $\le$ (i.e. a reflexive and transitive relation $\le$ with $x \le y \Rightarrow x + z \le y + z$) in the acting group G (for simplicity assume G abelian). One may say that we have "relativistic" time, similar to $G = R^4 =$ Space-Time of Special Relativity. A typical example is $\mathbf{Z}^d$ with the usual partial ordering (a d-tuple is ≥ 0 iff all the coordinates ≥ 0).

So assume G **Abelian and** $\le$ **a shift-invariant preordering in** G.

Define for $x, y \in G$:

$$[x, y] := \{z \in G | x \le z \le y\},]x, y[:= [x, y] \setminus \{x, y\}, [x, y[:= [x, y] \setminus \{y\},]x, y] := [x, y] \setminus \{x\}.$$

Definition 1.2.10 *Let $\omega \in \Omega$.*

The **return epoch (r.ep.)** *at ω is the set*
$\{x \in G : \omega \in E \ , \ x > 0 \ , \ x\omega \in E \ , \ \forall z \in]0, x[\ z\omega \notin E\}$
(where $y > x$ means $y \geq x \ , \ y \neq x$)
The **return duration (r.du.)** *at ω is the set*
$\{x \in G : \omega \in E \ , \ x \geq 0 \ , \ \forall z \in]0, x] \ z\omega \notin E\}$
Thus the return epoch and duration are empty for $\omega \notin E$.
The **arrival epoch (a.ep.)** *at ω is the set*
$\{x \in G : x \geq 0 \ , \ x\omega \in E \ , \ \forall z \in [0, x[\ z\omega \notin E\}$
The **arrival duration (a.du.)** *at ω is the set*
$\{x \in G : x \geq 0 \ , \ \forall z \in [0, x] \ z\omega \notin E\}$

Remark 1.2.11 For the **Z**-case, the return epoch is the singleton $\{\rho_E(\omega)\}$ for a.a. $\omega \in E$ and the arrival epoch is the singleton $\{\xi_E(\omega)\}$ for a.a. $\omega \in \mathit{satur}\, E$. The return duration and arrival duration are the intervals $[0, \rho_E(\omega)[$ and $[0, \xi_E(\omega)[$, resp.

In analogy with §1.2.1, apply HG to the following graphs depending on $\omega \in \Omega$ (fix $z \in G$):

$$F' = \{(x, y) \in G^2 : x - y = z, \ x\omega, y\omega \in E, \ \forall u \in]y, x[\ u\omega \notin E\}$$
$$F'' = \{(x, y) \in G^2 : x - y = z, \ \forall u \in [y, x] \ u\omega \notin E\}$$
$$F''' = \{(x, y) \in G^2 : x - y = z, \ y\omega \in E, \ \forall u \in]y, x] \ u\omega \notin E\}$$

One obtains the following version of Kac's for preordered time:

Proposition 1.2.12 *Consider the case of the acting group G being discrete countable preordered Abelian (with the preordering assumed, of course, shift-invariant). Let $E \subset \Omega$ be measurable.*

Consider the following pairs of sets, depending on ω

(i) the return epoch, and the return epoch w.r.t. the inverse action

(ii) the return duration, and the arrival epoch w.r.t. the inverse action

(iii) the arrival duration, and the arrival duration w.r.t. the inverse action

Then, for each of these pairs:
Every $z \in G$ has the same probability to belong to both members of the pair;
Consequently, the sums of a positive function $G \to \mathbf{R}^+$ on both members of the pair have the same expectation;
In particular, the cardinalities of both members of the pair have the same expectation.

QED

Kac's is the **Z**-case of the equality of the expectations of the cardinalities of the r.du. and the a.ep. for the inverse action.

Remark 1.2.13 In the "one-dimensional" **Z**-case, the durations determine the epochs and vice-versa (Remark 1.2.11), so we have complete "symmetry in expectation" between these attributes of the action and of the inverse (Prop. 1.2.1). In the partially-ordered case, there is no such symmetry for return durations (or arrival epochs), as is shown in the following example.

Example 1.2.14 Let $\mathbf{Z}^2$ (with the usual partial ordering) act on the "discrete torus" $(\mathbf{Z}/n\mathbf{Z})^2$ by addition modulo n.

Let E be the "upper triangle"

$$E = \left\{(s, t) \in \mathbf{Z}^2 : 1 \leq s \leq n, \ 1 \leq t \leq n, \ n \leq s + t\right\}$$

considered as a subset of $(\mathbf{Z}/n\mathbf{Z})^2$.

Let us find the cardinality $\rho(\omega)$ of the return duration at $\omega = (s,t)$, $1 \le s \le n$, $1 \le t \le n$ and the average of $\rho(\omega)$ over the "torus" (asymptotically as $n \to \infty$):
if $(s,t) \notin E$, $\quad \rho(\omega) = 0$;
if $(s,t) \in E$:
if $s < n$, $t < n$, $\quad \rho(\omega) = 1$, (the r.du. is $\{0\}$), adding to $\sim \frac{1}{2}n^2$;
if $s = n$, $t = n$, $\quad \rho(\omega) = 1$, (the r.du. is $\{0\}$);
if $s = n$, $t < n$, $\quad \rho(\omega) = n - t$ (the r.du. is a horizontal "segment"), adding to $\sim \frac{1}{2}n^2$;
similarly for $s < n$, $t = n$
Thus $\sum_\omega \rho(\omega) \sim \frac{3}{2}n^2$. The average is $\sim \frac{3}{2}$

Now let us do the same for $\rho^{(-)}$ – the return duration for the inverse action:
if $(s,t) \notin E$, $\quad \rho^{(-)}(\omega) = 0$;
if $(s,t) \in E$:
if $s + t > n$, $\quad \rho^{(-)}(\omega) = 1$, (the r.du. is $\{0\}$), adding to $\sim \frac{1}{2}n^2$;
if $s + t = n$, $\quad \rho^{(-)}(\omega) = st$ (the r.du. is a "rectangle"), adding to $\sim \frac{1}{6}n^3$;
and the average is $\sim \frac{1}{6}n$.

It might be interesting to estimate the averages of the cardinalities $\zeta(\omega)$, $\zeta^{(-)}(\omega)$ of the arrival epochs:
if $s + t \ge n$. $\quad \zeta(\omega) = 1$ (the a.ep. is $\{0\}$), adding to $\sim \frac{1}{2}n^2$;
if $s + t < n$, $\quad \zeta(\omega) = n - (s+t) + 1$ (the a.ep. is part of a "diagonal"), adding to $\sim \frac{1}{6}n^3$;
and the average is $\sim \frac{1}{6}n$.

For the inverse action:
if $s + t \ge n$, $\quad \zeta^{(-)}(\omega) = 1$ (the a.ep. is $\{0\}$), adding to $\sim \frac{1}{2}n^2$;
if $s + t < n$, $\quad \zeta^{(-)}(\omega) = 2$ (the a.ep. is composed of one point "below" and one point "to the left"), adding to $\sim n^2$;
and the average is $\sim \frac{2}{3}$
Thus our approximations agree with Prop. 1.2.12

2 The Continuous Case; Infinitesimal Measures

2.1 Introduction

For us, the "continuous" case means a Borel measure-preserving action of a 2nd-countable locally compact group[3] G on a standard measure space, i.e. a standard Borel space with a completion of a Borel measure (for comments about standard spaces see §A.2).[4] As a prototype one may think of such a *flow* – an $\mathbf{R}$-action on a standard probability space $(\Omega, \mathcal{B}, \mu)$. One may try to formulate a simple-minded generalization of Kac's formula, but as is well known, that would fail. For example, if Ω is the circle, our $\mathbf{R}$-action is rotation, and E is an interval, the return time is 0 except at one point, and its integral is 0.

We shall try to remedy this situation by letting μ induce an "infinitesimal measure" (in the above case this turns out to be the counting measure on the circle), and consider the integral of the return time w.r.t. this "infinitesimal measure" (in our example, this integral equals $1 - \mu(E)$). To do this and to formulate "continuous" VE, HG and "Kac" theorems, we apply the notion of invariant chains in the continuous case and use the multi-faceted way in which a function on Ω determines an invariant 0-chain.

Remark 2.1.1 Kac-like assertions in the "continuous" case appear in [He] (see §5.1) and even as early as [Bi] where one works with a "lower-dimensional" measure alongside the usual measure. Our "infinitesimal measure" has strong links to the Palm measures, standard in the theory of stationary random measures on a locally-compact group, in particular stationary point processes (see §2.5).

2.2 Chains in the Continuous Case

While simple-minded "Kac" fails in the continuous case, The "invariant chain" approach can be generalized, as follows: instead of *summing* over simplices, m-chains are formed by *integrating* over them w.r.t. a measure on G^{m+1}. Thus, while in the discrete case m-chains were, in fact, $m+1$-dimensional *matrices* depending on $\omega \in \Omega$, in the continuous case they will be *measures* ϕ over G^{m+1} again depending on $\omega \in \Omega$.

For instance, a simplex may be identified with the chain that is the δ-measure on a simplex, and a general chain may be obtained by integrating such entities over measures. (Here I deliberately ignore possible restrictions on the measures.)

Thus, in the discrete case, the chain given by $\sum_{k,l} F(k,l)\cdot(k,l)$ is viewed now as the measure $\sum_{k,l} F(k,l)\delta_{(k,l)}$ (on the discrete G^2).

In the sequel, a **measure** on a set X is merely an $\overline{\mathbf{R}^+} = [0,\infty]$-valued σ-additive function μ on a σ-algebra $\mathcal{B}$ of subsets of X (whose members are called "measurable sets"), with $\mu(\emptyset) = 0$. A **null set** is a member E of $\mathcal{B}$ with $\mu(E) = 0$. The measure is **complete** if every subset of a null set is a null set. Unless stated otherwise, measures are assumed complete.

If X is a 2nd-countable locally compact space, we shall usually assume that every Borel set is measurable, but we will not assume σ-finiteness nor that compact sets have finite measure. So, *our measures need not be Radon measures.*[5]

Returning to a G acting on Ω, The requirement for a chain (i.e. measure) $\phi(\omega)$ to be **invariant** is, *for unimodular* G, the analog of (7):

$$\forall y \in G \quad \phi(\omega) = \phi(y\omega) * \delta_{(y,\ldots,y)} \tag{14}$$

(recall: it is assumed that G acts on the left).

On the other hand, for general G we require, in view of the continuous VE Thm. in the next §, that the chain *share the right-invariance of the left Haar measure on* G. The left Haar measure λ on G satisfies:

$$\lambda = \Delta(y)\,\lambda * \delta_y$$

[3] This reference to topology is, in fact, inessential. A. Weil has shown (see [Ha-M] §59) that the structure of a 2nd-countable locally-compact topological group with Borel subsets and Haar measure can be equivalently defined purely measure-theoretically, as a Standard Measurable Group, i.e. a Group G which is also a standard Borel space with a σ-finite measure on the Borel sets, satisfying: $(x, y) \mapsto (x, xy)$ is a measure-preserving automorphism of $G \times G$ (in particular, the locally-compact topology in G can be defined as the weakest topology making all convolutions of two L^2 functions continuous).

[4] If a Polish topology is given in the standard space (compatible with the Borel structure) then a σ-finite complete measure on this Polish space is a completion of a Borel measure iff every open set is measurable and the measure is regular – see [Bo-I].

[5] On a 2nd-countable locally compact space, a measure is Radon iff it is finite on compact sets.

Where Δ is called the **modular function** of G and is a continuous homomorphism from G to the multiplicative group $\mathbf{R}^{+\times}$ (see [Loo]).

Thus we say that the chain ϕ is **Δ-right-invariant** if

$$\forall y \in G \quad \phi(\omega) = \Delta(y)\,\phi(y\omega) * \delta_{(y,\ldots,y)} \tag{15}$$

Clearly for uniomodular G this is just invariance (14).

A chain (i.e. measure) $\phi(\omega)$ depending on ω will be called **measurable**, if the integral of any *test function*, i.e. a nonnegative continuous function with compact support, w.r.t. $\phi(\omega)$ is measurable in ω. If these integrals are Borel-measurable in ω, the chain will be called **Borel-measurable**.

Also, the notion of **vertices** of chains generalizes readily: these are just the projections of the measures from G^{m+1} on the $m+1$ coordinates G. Indeed, they are obtained by integrating the δ-measure of the relevant vertex of the simplex w.r.t. the measure that the chain gives on the simplices (where integration of measure-valued functions may be defined via test-functions). Similarly, one may speak of lower-dimensional **faces** of an m-chain, which will be projections of the measure from G^{m+1} on some G^{k+1}, $k \le m$.

We may also speak about the **expectation** or **integral w.r.t.** ω of a measure $\phi(\omega)$ on G^{m+1} depending on ω: this will be the measure ε on G^{m+1} (now *not* a function of ω) s.t. the integral of any test-function i.e. nonnegative continuous function with compact support, w.r.t. ε is the expectation of its integral on $\phi(\omega)$. the issue of existence and uniqueness of this expectation will be made more precise when theorems are formulated. Since Radon measures on G^{m+1} are determined by their value on test functions, a Radon measure expectation is indeed unique. Such expectation will be referred to as a **Radon expectation** of the m-chain.

Note that the Radon expectation of a Δ-right-invariant 0-chain must be Δ-right-invariant, i.e. a *left Haar measure.*

If an Δ-right-invariant 0-chain has a Radon expectation, necessarily left Haar, one readily sees (by taking monotone increasing and bounded monotone decreasing limits with common compact support) that any non-negative Borel function on G will have the property of test functions, namely integrating it commutes with taking the expectation.

Remark 2.2.1 Note that one may choose a countable collection of test-functions – non-negative continuous with compact support – on G^{m+1} s.t. *every test-function is a non-decreasing limit of a sequence of members of $\mathcal{F}$.* (Take as $\mathcal{F}$, e.g., all positive rational combinations of the union of non-decreasing sequences of test-functions that converge to the characteristic functions of finite unions of members of a countable base to the topology in G^{m+1}.) Thus for our usual purposes it suffices to test on this countable collection of test-functions. Therefore if something about the measure (that depends, say, on ω) holds for a.a. ω for every fixed test-function it will hold for a.a. ω for the measure.

Note that we may impose on $\mathcal{F}$ that all its members are finite positive combinations of functions of the form $h_0(x_0)\cdots h_m(x_m)$, the h_i being test-functions on G (use the Stone-Weierstrass Approximation Theorem), or that all its members are finite positive combinations of convolutions of two test-functions. This is sometimes useful.

2.3 The Continuous VE (CVE) Theorem

In the previous § we extended the notion of chain from the discrete to the continuous case. Yet there are some important differences:

- In the continuous case, a hypergraph is not automatically a chain, since there is no "natural" measure on a set (except, of course, the highly massive "counting measure" which is usually not suitable).
- One cannot substitute in a measure, so the fact, holding in the discrete case, that e.g. any invariant 0-chain $\sum F_k(\omega)\cdot(k)$ comes from a function on Ω, namely $F_0(\omega)$, has no continuous analogue. In fact, as we shall see, *invariant* 0-*chains are quite richer than functions.*

These facts make the transition from a "VE" theorem to "HG" theorems more involved. Yet one may formulate readily a continuous VE theorem. In this theorem one does not need the assuption that (Ω, μ) is probability.

The Continuous Vertices Expectation (CVE) Theorem *Let a 2nd-countable locally compact group G act in a measure-preserving manner on a measure space $(\Omega, \mathcal{B}, \mu)$ (μ need not be σ-finite).*

Let us be given a Δ-right-invariant m-chain. This is a measure $\phi(\omega)$ on the set G^{m+1} of m-simplices, depending on ω (note it need not be a Radon measure), and assumed to be defined on Borel subsets of G^{m+1}. Assume the dependence of ϕ on ω is measurable, *i.e. the integral of a fixed test-function on G^{m+1} (that is, nonnegative continuous function with compact support) w.r.t. ϕ is measurable in ω. Δ-right-invariance means that (15) is satisfied.*

Suppose that one vertex of ϕ has a Radon μ-integral, equal to the (left Haar) measure λ on G. Then every vertex has the same Radon μ-integral λ.

Proof Since two vertices of an m-chain ϕ are vertices of some 1-chain "edge" of ϕ (i.e. projection of the measure ϕ on some G^2), and since measurability of the chain implies measurability of every "edge", we have to prove the assertion only for $m = 1$.

Let ϕ be our 1-chain. Thus, for $\omega \in \Omega$ $\phi(\omega)$ is a measure on G^2. Let $f : G \to \mathbf{R}$ be a test-function. Its integrals on the two vertices are the integrals w.r.t. ϕ of $(x, y) \mapsto f(x)$ and $(x, y) \mapsto f(y)$, which both depend on ω. We are told that, say,

$$\forall f \quad \int_\Omega \left(\int_{G^2} f(x)\, d\phi \right) d\mu(\omega) = \langle \lambda, f \rangle$$

and have to prove that

$$\forall f \quad \int_\Omega \left(\int_{G^2} f(y)\, d\phi \right) d\mu(\omega) = \langle \lambda, f \rangle\,.$$

Let U be a neighbourhood of $e \in G$, with $U^{-1} = U$. Let $(h_n(x))_n$ be a countable partition of unity on G (thus, $h_n \geq 0$, $\sum h_n(x) = 1$), where h_n are test-functions with $x, y \in \operatorname{supp} h_n \Rightarrow x^{-1}y \in U$.

The idea is to "slice" $(x, y) \mapsto f(x)$ (or, if one wishes, to slice $\phi(\omega)$) into diagonal slices using the partition of unity, and then to use the shift-invariance to move each slice diagonally so that the sum will approximate $(x, y) \mapsto f(y)$.

We have:

$$\langle \lambda, f \rangle = \int_\Omega \left(\int_{G^2} f(x)\, d\phi \right) d\mu(\omega) = \sum_n \int_\Omega \left(\int_{G^2} f(x) h_n(xy^{-1})\, d\phi \right) d\mu(\omega)$$

Denote by λ_n the functional on the test-functions f:

$$\langle \lambda_n, f \rangle := \int_\Omega \left(\int_{G^2} f(x) h_n(xy^{-1})\, d\phi \right) d\mu(\omega)$$

Recall that we have the Δ-right-invariance (15):

$$\forall a \in G \quad \phi(\omega) = \Delta(a)\, \phi(a\omega) * \delta_{(a,a)}$$

which means that for a function F on G^2

$$\int F(x_0, x_1)\, d(\phi(\omega))(x_0, x_1) = \Delta(a) \int F(x_0 a, x_1 a)\, d(\phi(a\omega))(x_0, x_1).$$

Thus for $a \in G$:

$$\begin{aligned}&\langle \lambda_n, f \rangle = \textstyle\int_\Omega \left(\int_{G^2} f(x) h_n(xy^{-1})\, d(\phi(\omega)) \right) d\mu(\omega) = \Delta(a) \int_\Omega \left(\int_{G^2} f(xa) h_n(xy^{-1})\, d(\phi(a\omega)) \right) d\mu(\omega) = \\ &= \Delta(a) \textstyle\int_\Omega \left(\int_{G^2} f(xa) h_n(xy^{-1})\, d(\phi(\omega)) \right) d\mu(\omega) = \Delta(a) \left\langle \lambda_n, (x \mapsto f(xa)) \right\rangle\end{aligned}$$

Thus λ_n is Δ-right-invariant. Since it is positive and finite on test-functions (being dominated by λ), it is a left Haar measure. Thus, for $x_n \in G$, to be chosen later, we have

$$\begin{aligned}&\langle \lambda, f \rangle = \textstyle\sum_n \int_\Omega \left(\int_{G^2} f(x_n x) h_n(xy^{-1})\, d\phi \right) d\mu(\omega) = \\ &= \textstyle\int_\Omega \left(\int_{G^2} \left[f(y) + \sum_n (f(x_n x) - f(y)) h_n(xy^{-1}) \right] d\phi \right) d\mu(\omega)\end{aligned}$$

For a fixed test-function f, we have to estimate the error:

$$\begin{aligned}\mathcal{E} &= \int_\Omega\bigl(\int_{G^2} f(x)\,d\phi\bigr)\,d\mu(\omega) - \int_\Omega\bigl(\int_{G^2} f(y)\,d\phi\bigr)\,d\mu(\omega) = \\ &= \int_\Omega\left(\int_{G^2}\left[\sum_n (f(x_n x) - f(y))h_n(xy^{-1})\right]\,d\phi\right)\,d\mu(\omega)\end{aligned}$$

To this end (f is fixed, fix $\varepsilon > 0$ and fix a relatively compact e-neighbourhood U_0), choose the above $U = U^{-1} \subset U_0$ so that $xy^{-1} \in U \Rightarrow |f(x) - f(y)| < \varepsilon$ and choose $x_n^{-1} \in \operatorname{supp} h_n$. If $xy^{-1} \in \operatorname{supp} h_n$ then $x_n xy^{-1} \in U$, hence $|f(x_n x) - f(y)| < \varepsilon$. Thus, if k is a continuous nonnegative function with compact support s.t. $k \geq 1$ in $U_0 \cdot (\operatorname{supp} f)$, then $xy^{-1} \in \operatorname{supp} h_n \Rightarrow |f(x_n x) - f(y)| \leq \varepsilon k(x_n x)$, hence, using the left Haar'ness of λ_n:

$$\begin{aligned}|\mathcal{E}| &\leq \int_\Omega\left(\int_{G^2}\left[\sum_n |f(x_n x) - f(y)| h_n(xy^{-1})\right]\,d\phi\right)\,d\mu(\omega) \leq \\ &\leq \varepsilon \int_\Omega\left(\int_{G^2}\left[\sum_n k(x_n x) h_n(xy^{-1})\right]\,d\phi\right)\,d\mu(\omega) = \\ &= \varepsilon \int_\Omega\left(\int_{G^2}\left[\sum_n k(x) h_n(xy^{-1})\right]\,d\phi\right)\,d\mu(\omega) \leq \varepsilon\,\langle\lambda, k\rangle\end{aligned}$$

Since $\mathcal{E}$ does not depend on ε, one concludes that $\mathcal{E} = 0$ and we are done.

QED

Remark 2.3.1 Note that a statement like the CVE Thm. cannot hold for chains with other kind of "right-invariance", i.e. w.r.t. a homomorphism from G to $\mathbf{R}^{+\times}$ other than Δ. Indeed, such invariance would be inherited by the vertices and their expectations, and if a statement such as CVE holds, these expectations must be left-invariant. That follows from the fact that replacing a 1-chain ϕ by $\delta_{(0,a)} * \phi$ preserves its kind of right-invariance, does not change one vertex but left-shifts the other.

2.4 Enhanced Functions and Infinitesimal Measures

For the rest of Section 2 we place ourself in the setting of a locally compact group G acting in a Borel and measure-preserving manner on a standard σ-finite measure space $(\Omega, \mathcal{B}, \mu)$. Although we need not assume that the measure μ is probability, we still speak of "expectation" instead of μ-integral, since our main interest lies in the probability case. Denote a left Haar measure in G by λ.

As already mentioned, unlike the discrete case, there is no 1-1 correspondence between measurable functions on Ω and Δ-right-invariant 0-chains. Given a (nonnegative) measurable function f on Ω, one can still correspond to it a measurable Δ-right-invariant 0-chain, mapping each $\omega \in \Omega$ to $f(x\omega)\,d\lambda(x)$. To check that this 0-chain is indeed Δ-right-invariant, that is, satisfies (15):

$$\Delta(y)\ [f(x \cdot y \cdot \omega)\,d\lambda(x)] * \delta_y = f(x \cdot \omega)\,d\lambda(x). \tag{16}$$

Its expectation is $\mathbf{E}(f)\lambda$, as one finds using Fubini: indeed, if $h : G \to \mathbf{R}$ is a test-function,

$$\begin{aligned}&\int_\Omega\left[\int_G h(x)f(x\omega)\,d\lambda(x)\right]\,d\mu(\omega) = \\ &= \int_G\left[\int_\Omega h(x)f(x\omega)\,d\mu(\omega)\right]\,d\lambda(x) = \\ &= \int_G\left[\int_\Omega h(x)f(\omega)\,d\mu(\omega)\right]\,d\lambda(x) = \int_G h(x)\mathbf{E}(f)\,d\lambda(x),\end{aligned}$$

and we sometimes identify this 0-chain with the "ordinary" function f.

But let Λ be some other Δ-right-invariant measure (as usual, assumed complete, but not necessarily σ-finite) on G, i.e. satisfying

$$\forall y \in G \quad \Lambda = \Delta(y)\,\Lambda * \delta_y \tag{17}$$

(In the case of unimodular G this is just right-invariance), s.t. Borel sets are Λ-measurable. An example is the counting measure, in the case of unimodular G. Let $f : \Omega \to \overline{R^+} = [0, \infty]$. Suppose that for a.a. ω $x \mapsto f(x\omega)$ is Λ-measurable. One may consider the 0-chain mapping each $\omega \in \Omega$ to the measure on G $f(x\omega)\,d\Lambda(x)$. This 0-chain will be denoted by $f\frac{d\Lambda}{d}$ (the rationale behind this notation will be seen below). Suppose f *is s.t.* $f\frac{d\Lambda}{d}$ *is measurable.* It will be Δ-right-invariant *for the same reason as before*, (16). And *even if f has expectation* 0, $f\frac{d\Lambda}{d}$ may have an expectation $a\lambda$, $a \geq 0$. Here $a > 0$ is possible since *Fubini need not hold for $\mu(\omega), \Lambda(x)$.* We shall write then

$$\mathbf{E}\left(f\frac{d\Lambda}{d\lambda}\right) = a \tag{18}$$

and view the expression $f\frac{d\Lambda}{d\lambda}$ as an **enhanced function**, f being "enhanced" by the *"infinite constant* $\frac{d\Lambda}{d\lambda}$*"*. (This mock-"Radon-Nikodym derivative" should indeed be viewed as a "constant" since both λ and Λ are Δ-right-invariant.) This notation is further justified by the fact that if Λ happens to be Radon, hence some (other) left Haar, $\frac{d\Lambda}{d\lambda}$ is just a number multiplier in the above.

When the Haar measure λ is fixed, we identify $f\frac{d\Lambda}{d\lambda}$ with the 0-chain $f\frac{d\Lambda}{d}$ and speak of an enhanced function as a special case of 0-chain.

To be more precise, one may think of an enhanced function as a positively homogeneous mapping from the half-line of left Haar measures on G to the cone of non-negative Δ-right-invariant (ω-dependent) 0-chains, denoted by $f\frac{d\Lambda}{d\lambda}$ if it maps $\lambda \mapsto f\frac{d\Lambda}{d}$. Such entities can be added and multiplied by non-negative constants and by functions f on Ω (i.e. by $(x;\omega) \mapsto f(x\omega)$). The *"infinite constants"* such as $\frac{d\Lambda}{d\lambda}$ are to be defined as positively homogeneous maps from the half-line of left Haar measures on G to the cone of (not necessarily Radon) Δ-right-invariant measures in G.

Note that (18) means just that for *arbitrary* test-function $h : G \to \mathbf{R}^+$ (here we may take, in our case, any nonnegative Borel function) we have:

$$\int_\Omega \left(\int_G h(x)f(x\omega)\,d\Lambda(x)\right)\,d\mu(\omega) = a\int_G h(x)\,d\lambda(x) \tag{19}$$

Note that by invariance we are sure that if the enhanced f has a Radon expectation at all, it will be a multiple of λ so the left-hand side of (19) is a constant multiple of $\int_G h\,d\lambda$.

Remark 2.4.1 For example, for $\mathbf{R}$-action (18) means just that for $\omega \in \Omega$, if we consider the Λ integral on Time of f over an interval in the past or future, and take the expectation for ω, we get a times the λ-length of that interval.

Remark 2.4.2 Note that if f has a 0-chain "enhancement" which is *measurable* and (19) holds for some finite $a \geq 0$ and *one* nonnegative integrable Borel h which is bounded below away from 0 on an open set, then the "enhancement" has the expectation $a\,d\lambda$, hence (19) holds for *every* nonnegative Borel function i.e. (18) holds. (by invariance (19) holds for all combinations of translates of h which dominate any continuous with compact support, hence the expectation is Radon.)

Remark 2.4.3 In fact, the notation (18) is somewhat misleading – it conceals the fact that these notions depend on the particular action of the group G on Ω.

In order to clarify what was said, consider the following example alluded to before:

Example 2.4.4 Take $G = \mathbf{R}$, $\Omega =$ the circle $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ with Lebesgue probability measure, $\mathbf{R}$ acting by rotation. Let $E =$ a finite union of closed intervals in $\mathbf{T}$. Let f be the simple-minded *return function* to E. It is 0 except for the finite set of the upper extremities of the intervals (where the motion "exits" from E). Of course, f has zero expectation. But consider in $\mathbf{R}$ the Haar $\lambda = dx$ and the invariant $\Lambda =$ count $=$ count$_{\mathbf{R}}$ – the counting measure. One easily shows that for any $f : \mathbf{T} \to \overline{R^+}$ with countable support, $\mathbf{E}\left(f\frac{d\text{count}}{dx}\right) = \sum_t f(t)$. Thus, "enhanced" by $\frac{d\text{count}}{dx}$ our return function has always the expectation $1 - \mu(E)$ which is an instance of a "Kac" theorem to be proved later (Thm. 2.8.1)

As seen in the last example, taking the expectation of an "enhancement" of functions f on Ω may amount to integrating f on some measure on Ω (count$_{\mathbf{T}}$ in the example) that gives positive mass to μ-null sets, yet is "induced" by μ (and Λ, λ). Call it an **infinitesimal measure**, since it may be thought of as specifying the "infinitesimal mass" of a set $\subset \Omega$. Denote it by $\frac{d\Lambda}{d\lambda}\mu$. ($\mu$ is "enhanced" by "multiplication" by the "constant" $\frac{d\Lambda}{d\lambda}$.)

In order to consider such "infinitesimal measures" in the general setting, proceed as follows:

We use the generation of a measure via "preintegrable" functions as explained in §A.1. The set Pre of *preintegrable* functions f will consist of the $[0,\infty]$-valued functions which have an enhancement which is a measurable 0-chain with a Radon expectation, thus the "enhanced" f has a finite expectation, which will be the *integral* of f.

One checks easily that the axioms 1-4 in §A.1 are satisfied (recall Λ is assumed complete). Thus the "infinitesimal measure" $\frac{d\Lambda}{d\lambda}\mu$ is defined on some σ-algebra of subsets of Ω. We have:

Proposition 2.4.5 *Every Borel subset of Ω is measurable w.r.t. the "infinitesimal measure" $\frac{d\Lambda}{d\lambda}\mu$.*

Proof We refer to §A.3. One may assume Ω is an invariant Borel subset of a compact metrizable G-space $\bar{\Omega}$, where the action of G on $\bar{\Omega}$ is continuous in the two variables. It suffices to prove the assertrion for $\bar{\Omega}$ instead of Ω. Note that μ on $\bar{\Omega}$ is not neccessarily finite thus not neccessarily Radon.

One needs to prove that for every *closed* $K \subset \bar{\Omega}$, K is $\frac{d\Lambda}{d\lambda}\mu$-measurable, i.e. (see §A.1) that $\forall f \in \mathrm{Pre}\,,\ f \cdot 1_K \in \mathrm{Pre}$. One needs to know that if the 0-chain $f(x\omega)\,d\Lambda$ is measurable with Radon expectation, so is $f(x\omega)1_K(x\omega)\,d\Lambda$. In fact, this will hold for $1_{K'}(x,\omega)$ for any closed $K' \subset G\times\bar{\Omega}$, instead of $1_K(x\omega)$. That follows from its holding for such closed K' which are finite unions of "rectangles" of the form closed $\times$ compact $\subset G\times\bar{\Omega}$, which have a general closed K' as a countable decreasing intersection (note that we are always checking test-functions with *compact* support in G). For the latter the measurability and having Radon expectation are evident.

QED

Note that (see §A.1) for a $\frac{d\Lambda}{d\lambda}\mu$-measurable nonnegative f, in particular for nonnegative Borel functions f, f has a finite $\frac{d\Lambda}{d\lambda}\mu$-integral iff it is in Pre i.e. iff it has an "enhancement" which is a measurable 0-chain with Radon expectation, its integral being equal to that expectation.

The following proposition is easily proved:

Proposition 2.4.6 *Suppose Λ_1 and Λ_2 are Δ-right-invariant measures on G s.t. every set of finite Λ_1 mass has zero Λ_2 mass. Then every $\frac{d\Lambda_1}{d\lambda}\mu$ -integrable function has zero $\frac{d\Lambda_2}{d\lambda}\mu$ -integral.*

Example 2.4.7 of "infinitesimal measures":

1. A situation which includes Exm. 2.4.4. G – some unimodular Lie group with Haar λ, Γ – a discrete subgroup s.t. λ induces on the homogeneous space $\Omega = G/\Gamma$ an (invariant) probability measure μ. G acts on Ω by left multiplication. Λ – some right-invariant measure in G.

   Using the test-function 1_D where D is a fundamental domain for Γ, one concludes that $\frac{d\Lambda}{d\lambda}\mu$ is just the measure on Ω transported from $\Lambda|_D$ by $(x \mapsto x\omega) : G \to \Omega$.

2. Let $\mathbf{R}$ act on the torus $\mathbf{T}^2 = (\mathbf{R}/\mathbf{Z})^2$ (with Lebesgue measure μ) by $x(s,t) = (s + ax, t + bx)$, $x \in \mathbf{R}$. When a/b is *irrational*, this is ergodic. Let $\lambda = dx, \Lambda = \mathrm{count}_{\mathbf{R}}$. $\frac{d\mathrm{count}}{dx}\mu$ measures "the area swept by a set in unit time", i.e. $\frac{d\mathrm{count}}{dx}\mu(E)$ is the area swept by E during some interval of time, with multiplicity, divided by the length of the interval. It follows from (20) below that for a smooth curve E in $\mathbf{T}^2$,

   $$\frac{d\mathrm{count}}{dx}\mu(E) = \int_E |-b\,ds + a\,dt|$$

   The latter differential form being the contraction of the area 2-differential in $\Omega = \mathbf{T}^2$ by the vector-field induced on Ω by $\frac{\partial}{\partial x}$ on $\mathbf{R}$.

3. Similarly, let an n-dimensional Lie group G with left Haar measure λ act smoothly and measure-preservingly (on the left) on a smooth m-dimensional manifold Ω with (invariant) probability measure μ given by an m-differential form, also donoted by μ (actually by the absolute value of this differential form, so that orientations do not matter), Let Λ be an Δ-right-invariant measure in G given on submanifolds by (the absolute value of) a Δ-right-invariant k-differential form, also denoted by Λ. The left Haar measure λ is given by (the absolute value of) a left-invariant n-differential form, to be denoted also by λ. Then one has the following formula:

   Let γ be an n-vector in the tangent space T_eG s.t. $|\langle\lambda,\gamma\rangle| = 1$. Then $\frac{d\Lambda}{d\lambda}\mu$ is given on submanifolds by (the absolute value of) the ℓ-differential form, $\ell = m + k - n$

   $$(\Lambda\lfloor\gamma)\,\omega\lfloor\mu = ((\omega,u) \mapsto \langle\mu, (\langle\Lambda, c\gamma\rangle)\,\omega \wedge u\rangle) \tag{20}$$

where $\lfloor$ denotes contraction, c is the coproduct, u is a variable ℓ-vector in $T_\omega\Omega$ and applying something in the tangent space T_eG to ω means applying the derivative of $(x \mapsto x\omega)$ at e to the "something".

To prove (20), let E be an ℓ-dimensional submanifold in Ω. Let $\phi : G \times \Omega \to G \times \Omega$ be given by $\phi(x, \omega) = (x, x\omega)$, thus $\phi^{-1}(x, \omega) = (x, x^{-1}\omega)$. By (18) and (19), we compute $\frac{d\Lambda}{d\lambda}\mu(E)$ as follows:

Construct the $\ell + n = m + k$ -submanifold of $G \times \Omega$ $\tilde{E} := \phi^{-1}(G \times E)$, with the projection $\pi : \tilde{E} \to \Omega$ given by $\pi(x, \omega) = \omega$ (with fibers $\pi^{-1}(\omega)$ that we sometimes identify with subsets of G). By Sard's Lemma (see [Sch] Ch. I), for a.a. ω π is a submersion throughout the fiber $\pi^{-1}(\omega)$ so that the fiber is a k-manifold in G. To compute $\frac{d\Lambda}{d\lambda}\mu(E)$, choose an open $U \subset G$ with $0 < \lambda(U) < \infty$, for each ω integrate Λ on $\pi^{-1}(\omega) \cap U$, then integrate on $d\mu(\omega)$ and divide by $\lambda(U)$. This iterated integration is given by integration on $\tilde{E} \cap (U \times \Omega)$ of the (absolute value of) the $(m + k)$-differential form ν in $G \times \Omega$ given by $(\xi_i, \xi'_i \in T_xG,\ v_i \in T_\omega\Omega)$:

$$\langle \nu, (\xi_1, 0) \wedge \ldots \wedge (\xi_k, 0) \wedge (\xi'_1, v_1) \wedge \ldots \wedge (\xi'_m, v_m)\rangle = \langle \Lambda, \xi_1 \wedge \ldots \wedge \xi_k\rangle\, \langle \mu, v_1 \wedge \ldots \wedge v_m\rangle \tag{21}$$

(Note that at points $(x, \omega) \in \tilde{E}$ where π is not a submersion, i.e. its derivative is not onto $T_\omega\Omega$, $\nu|\tilde{E}$ vanishes.)

To proceed, the measure $\frac{d\Lambda}{d\lambda}\mu$ on E will be given by an ℓ-differential form s.t. integration w.r.t. ν will be the result of another iterated integration, here using the projection $\pi' : \tilde{E} \to E$ given by $\pi'(x, \omega) = x\omega$ and taking the integration w.r.t. λ on the fibers (identified with G by $x \mapsto \phi^{-1}(x, \omega) = (x, x^{-1}\omega)$) and then integrating w.r.t. that ℓ-differential form on E. This means that for $v_1, \ldots, v_\ell \in T_\omega E$, taking $x = e$ and writing the above γ as $\gamma = \xi_1 \wedge \ldots \wedge \xi_n$, $\xi_i \in T_eG$ (note that the above identification $x \mapsto (x, x^{-1}\omega)$ gives, on taking derivatives, $\xi_i \mapsto (\xi_i, -\xi_i\omega)$), the following ℓ-differential form will do:

$$\left\langle \frac{d\Lambda}{d\lambda}\mu, v_1 \wedge \ldots \wedge v_\ell \right\rangle = \langle \nu, (0, v_1) \wedge \ldots \wedge (0, v_\ell) \wedge (\xi_1, -\xi_1\omega) \wedge \ldots \wedge (\xi_n, -\xi_n\omega)\rangle \tag{22}$$

And (20) follows from (21) and (22).

See [Fed] for further pertinent theory about measures defined by differential forms.

4. Yet one must be warned that "infinitesimal measures" may behave strangely: Take, for instance $G = \mathbf{R}^2$ acting on the circle $\Omega = \mathbf{T} = \mathbf{R}/\mathbf{Z}$, endowed with Lebesgue μ, by rotation using one coordinate: $(x, y)(t) = t + x \in \mathbf{T}$, $(x, y) \in \mathbf{R}^2$. Consider some cases for Λ (to compute the infinitesimal measures, take as test-function the characteristic function of a fundamental domain of $\mathbf{Z}^2$ in $\mathbf{R}^2$):

   For $\Lambda = \mathrm{count}_G$, $\frac{d\Lambda}{dx\,dy}\mu = \infty \cdot \mathrm{count}$, which gives mass ∞ to any set except $\emptyset$.

   For Λ an invariant measure which, reduced to any coset of some fixed 1-dimensional subspace $H \subset G$, is Lebesgue on the coset: if $H = \{(0, y) : y \in \mathbf{R}\}$ then $\frac{d\Lambda}{dx\,dy}\mu = \mathrm{count}_\Omega$. If H is otherwise, i.e. "slanted", then if a set in G that intersects an uncountable number of H-cosets has Λ-measure ∞, then $\frac{d\Lambda}{dx\,dy}\mu = \infty \cdot \mathrm{count}$ If Λ is the sum of Lebesgue measures of the intersections with all H-cosets, then $\frac{d\Lambda}{dx\,dy}\mu = \infty \cdot \mu$ where $\infty \cdot \mu$ gives mass 0 to Lebesgue-null subsets of Ω, and mass ∞ to sets with positive Lebesgue mass.

   For Λ with mass of a set equal to the integral of the numbers of points of intersection of the set with the H-cosets, w.r.t. some Haar on G/H (on smooth curves it will be given by some 1-differential $a\,dx + b\,dy$ – see [Fed]): If $H = \{(0, y) : y \in \mathbf{R}\}$ then $\frac{d\Lambda}{dx\,dy}\mu = \infty \cdot \mu$. Otherwise $\frac{d\Lambda}{dx\,dy}\mu$ is a multiple of count_Ω.

5. Further to the previous item, let $SO(3)$ act on the sphere S^2 by usual rotations, λ the normalized Haar on $SO(3)$ and μ normalized (i.e. probabilty) invariant area on S^2. This example has in common with the previous one the property that the stabilizer of any point is subgroup of positive dimension.

   Indeed, here $\frac{d\mathrm{count}}{d\lambda}\mu$ is $\infty \cdot \mathrm{count}$ (take as test-function the constant 1).

6. "Infinitesimal measures", restricted to important subsets $E \subset \Omega$ of μ-mass 0 provide interesting measure spaces, generalizing the usual $\mu|_E$ when $\mu(E) > 0$.

   One may think of $\Omega =$ the set of discrete subsets ω of $\mathbf{R}^n$ with μ given by the Poisson distribution corresponding to a Haar measure λ on $\mathbf{R}^n$, i.e. s.t. the expectation, for a Borel $A \subset \mathbf{R}^n$, of the number $\#(\omega \cup A)$ is $\lambda(A)$. Let $\mathbf{R}^n$ act on Ω by shift, and let $E = \{\omega : 0 \in \omega\}$. Taking as a test-function a characteristic function of some bounded domain in R^n, one finds:

   $$\frac{d\text{count}}{d\lambda}\mu(E) = 1$$

   This is a case of Palm measure (see §2.5).

   More can be said on this in view of §2.8.

   Another example is $\Omega = \mathcal{C}(\mathbf{R})/$the constants with $\mu =$ Brownian motion and $\mathbf{R}$ acting by shift, and $E =$ those motions that return at 1 to where they had been at 0. One may guess that for this E taking as Λ some Hausdorff measure might be interesting, but I have not thought on that.

7. Suppose $\chi : \Omega \to \mathbf{R}$ is some stochastic variable. One may consider $\mathbf{E}\frac{d\text{count}}{d\lambda}\{\chi = a\} =$ the "infinitesimal expectation that $\chi = a$". Curiously enough, in general this *cannot* be integrated on a to give $\mathbf{E}\chi$, as can be seen considering χ a smooth stochastic variable on T^2 acted by $\mathbf{R}$ as in item 2. Moreover, this depends on the action of G. Some such situations will appear in §2.8.

2.5 Links with Palm measures

The notion of Palm measure is standard in the theory of stationary random measures on locally compact groups, in particular stationary point processes (see [Me], [De], [Ne-P], [DV], [Ne]). In our context, Palm measures are measures on Ω with enhanced functions (or more general invariant 0-chains) as densities.

For us, an invariant 0-chain, which in the discrete case is an equivalent way to give a function, in the continuous case played the role of a generalized function. But a 0-chain can be viewed also as a stationary measure-valued stochastic variable, describing a stochastic random measure on G. To such an object one associates a Palm measure on Ω. To describe the Palm measure from our point of view, note that if (Ω, μ) is a Borel measure space acted in a Borel manner by, say, an abelian locally compact group G, a measurable function $f : \Omega \to \overline{\mathbf{R}^+}$ defines a measure $f\, d\mu$ on Ω with f as density. Viewing f as a 0-chain $\Phi : \omega \mapsto f(x\omega)d\lambda(x)$ (where λ is a Haar measure on G), one computes the integral of a non-negative Borel function g w.r.t. $f\, d\mu$ by multiplying the 0-chain Φ by $g(x\omega)$ and taking the "expectation" of the resulting 0-chain. This means that if $h : G \to \overline{\mathbf{R}^+}$ is Borel, one has:

$$\langle f\, d\mu, g\rangle \int h\, d\lambda = \int d\mu(\omega) \left(\int h(x) g(x\omega) f(x\omega)\, d\lambda(x) \right)$$

and this can be generalized to any Borel-measurable invariant 0-chain Φ to define the Borel measure $\Phi\, d\mu$ on Ω with density Φ, called the Palm measure:

$$\langle \Phi\, d\mu, g\rangle \int h\, d\lambda = \int d\mu(\omega) \left(h(x) g(x\omega)\ \left(d\Phi(\omega)\right)(x)\right).$$

In particular, if Λ is an invariant measure on G (not necessarily σ-finite) and $E \subset \Omega$ is Borel, then the enhanced function $1_E \frac{d\Lambda}{d\lambda}$ has a Palm measure $1_E \frac{d\Lambda}{d\lambda}\, d\mu$ which is just the restriction of the "infinitesimal measure" $\frac{d\Lambda}{d\lambda}\, d\mu$ to E. If $\mathcal{O}_\omega E$ is discrete for a.a. ω, then this random discrete set defines a stationary point process with this Palm measure. This is the situation in Ex. 2.4.7 6.

2.6 Hypergraphs, Weighted Hypergraphs and the Continuous HG (CHG) Theorem

We restrict ourselves to *unimodular* acting group G.

As mentioned above, in the continuous case **hypergraphs** do not automatically define m-chains. One may also consider (ω-dependent) nonnegative functions on the set of m-simplices ("matrices"), now *not* the same as m-chains (they do not have vertices). Such functions will be referred to as **weighted hypergraphs** $F(x_0, \ldots, x_m; \omega)$ $x_i \in G, \omega \in \Omega$. They will be assumed **(right)-invariant**, in the sense that:

$$F(x_0 y, \ldots, x_m y; \omega) = F(x_0, \ldots, x_m; y\omega) \quad x_i, y \in G, \omega \in \Omega \tag{23}$$

Such a weighted hypergraph may be converted into an m-chain if an *m-simplex* $(\Lambda_0, \ldots, \Lambda_m)$ *of right-invariant measures in* G is given. Assuming enough measurability and "Fubini", just multiply F, for each ω, by the *product measure* on G^{m+1}.

CVE leads to the following:

The Continuous Hypergraph (CHG) Theorem. *Let a 2nd-countable* unimodular *locally compact group G act in a Borel and measure-preserving manner on a standard σ-finite measure space $(\Omega, \mathcal{B}, \mu)$. Let λ be a Haar measure on G.*

Let us be given a (right-)invariant weighted hypergraph. This is a nonnegative function $F(x_0, \ldots, x_m; \omega)$ on the set G^{m+1} of m-simplices, depending on ω. (Right-)invariance means that (23) is satisfied.

Let us be given also an m-simplex $(\Lambda_0, \ldots, \Lambda_m)$ of right-invariant complete measures in G for which Borel sets are measurable (they need not be a Radon measures).

Assume:

- *For a.a. ω, F on G^{m+1} is measurable w.r.t. to the product measure, and is* 0 *outside a countable union of products in G^{m+1} of sets with finite corresponding Λ_i-measures, so Fubini holds for the product of measures.*
- *When F is multiplied by that product measure, the invariant m-chain obtained is measurable.*

Then for all $i = 0, \ldots, m$, the integral of the weighted hypergraph F, over the set of simplices with 0 *as the i-th vertex, w.r.t. the product of the measures $\Lambda_{i'}$, $i' \neq i$, this integral being a function of ω, is $\frac{d\Lambda_i}{d\lambda}\mu$-measurable. If, for some $i = 0, \ldots, m$, it has, when enhanced by $\frac{d\Lambda_i}{d\lambda}$, a finite expectation (that is, μ-integral), then the same holds for any i, with the same expectation (and since the only other possibility is all these expectations being ∞, the word "finite" may be deleted).*

The above ω-depending integral, or its above enhancement by $\frac{d\Lambda_i}{d\lambda}$, will be called: **the i'th vertex of the weighted hypergraph w.r.t. the simplex of measures**.

Proof Note that *we do not have Fubini* for interchanging integrations w.r.t. Λ_i and μ, but the fact that we have Fubini for the Λ_i's defines the product measure unequivocally.

We formulate the proof for $m = 1$, applying CVE to the 1-chain. We have to consider its two vertices, i.e. projections on G. These are:

$$\begin{array}{lcl} \text{source} & = & \omega \mapsto \left[\int_G F(x_0, x_1; \omega)\, d\Lambda_1(x_1)\right]\, d\Lambda_0(x_0) \\ \text{target} & = & \omega \mapsto \left[\int_G F(x_0, x_1; \omega)\, d\Lambda_0(x_0)\right]\, d\Lambda_1(x_1) \end{array}$$

(Here Fubini for Λ_0 and Λ_1 was used implicitly.)

The source is a Λ_0-enhancement of the function

$$\omega \mapsto \int_G F(0, x_1; \omega)\, d\Lambda_1(x_1).$$

that follows from the fact that substituting $x_0\omega$ for ω in this function gives

$$\int_G F(0, x_1; x_0\omega)\, d\Lambda_1(x_1) = \int_G F(x_0, x_1 x_0; \omega)\, d\Lambda_1(x_1) = \int_G F(x_0, x_1; \omega)\, d\Lambda_1(x_1)$$

(here we need actual right-invariance of Λ_1, hence unimodularity), similarly for the target. These are the enhanced functions mentioned in the theorem. If one of these has a finite expectation, then the 0-chain vertex has a Radon expectation and applying CVE we are done.

QED

It might be helpful to try to find conditions easier to check than measurability in the CHG Thm.

2.7 The Question of Measurability

To apply CHG, one needs to know that the m-chain is measurable.

We refer to §A.2.

In view of following sections (such as Section 4) we wish not to refer to a particular (probability) measure μ in Ω. This gives us the option to require Borel-measurability. Another significant notion is sets or functions being **universally measurable** – measurable w.r.t. any completion of a finite (equivalently, σ-finite) Borel measure. It is known (see [Ke-D] Thm. (21.10) or [Ku] §11 VII. or [Bo-T] Ch. IX §6) that any Suslin set in a standard space, i.e. any image of some (Borel subset of a) standard space by a Borel mapping, is universally measurable. (If the Borel mapping is countable-to-one, i.e. the preimage of every point is at most countable, then the image is Borel – see [Lu] Ch. III,IV).

Note that if a function $f : X \to Y$ between standard Borel spaces is given, and a Lusin topology is given in each of the spaces, then f is universally measurable iff for any finite measure in X s.t. Borel sets are measurable, there is a compact $K \subset X$ with K^c of mass as small as we please, s.t. $f|_K$ is continuous.

Consequently, the composition of universally measurable mappings between standard Borel spaces is universally measurable.

In many of our applications the measurability of the m-chain in an application of CHG may be assured by constructing the (weighted) hypergraph in two stages:

First, one corresponds to each ω a *closed subset* $F(\omega) \subset G$. In many cases this will be the closure $\overline{\mathcal{O}_\omega E}$ for some Borel $E \subset \Omega$. Another alternative is the *essential closure* of $\mathcal{O}_\omega E$ – the set of points in G no neighbourhood of which intersects $\mathcal{O}_\omega E$ in a Haar-null set. In §5.1 other alternatives are considered.

Second, the (weighted) hypergraph dependence on ω is obtained by a rule corresponding to each closed subset of G a (weighted) hypergraph, with no mention of Ω.

Now in the set of closed subsets of G we always take the Effros Borel structure (see [Ke-D]) – just identify each closed set F with the set of members u of a fixed countable open base to the topology that satisfy $u \cap F = \emptyset$, that set being a member of $2^{\mathbf{N}}$. This is a standard Borel space (This Borel structure can be given in many other ways, e.g. it is the Borel structure of the topological space 2^G (with members the *closed* subsets of G) with subbasis consisting of sets of all closed set contained, or all closed sets intersecting, an open set – see [Ku]. This topology is given by the Hausdorff metric for a restriction of any metric of the Alexandrov (one-point) compactification of G.)

In the cases mentioned above one is sure that $\omega \mapsto F(\omega)$ is measurable. Indeed, for $F(\omega) = \overline{\mathcal{O}_\omega E}$, one has:

Proposition 2.7.1 *Let a 2nd countable locally compact group G act in a Borel manner on a standard Borel space Ω. Let $E \subset \Omega$ be Borel. Then:*

(i) The mapping: $\omega \mapsto \overline{\mathcal{O}_\omega E}$ from Ω to the Borel space of closed subsets of G is universally measurable.

(ii) (see §A.3) Suppose $\mathcal{O}_\omega E$ is countable *for every $\omega \in \Omega$. Then there exists an embedding of the G-Borel space Ω in a G-metrizable compact space and a $E' \subset \Omega$ which is G_δ in the relative topology s.t. $\forall\, \omega \in \Omega$, $\overline{\mathcal{O}_\omega E} = \mathcal{O}_\omega E'$.*

(iii) Suppose $\mathcal{O}_\omega E$ is countable *for every $\omega \in \Omega$. Then the mapping: $\omega \mapsto \overline{\mathcal{O}_\omega E}$ from Ω to the Borel space of closed subsets of G is Borel.*

Proof (i) By the way the Borel structure in the set of closed sets is defined, it suffices that for a fixed open $u \subset G$, the set of ω s.t. u intersects $\overline{\mathcal{O}_\omega E}$, that is, u intersects $\mathcal{O}_\omega E$, is universally measurable. But this set is Suslin, being a projection of a Borel set $\subset \Omega \times G$.

(ii) Let λ be a right Haar measure in G. Choose a sequence h_n of $\infty\times$the characteristic functions of a decreasing sequence U_n of open neighbourhoods of 0 in G which forms a basis to the neighbourhoods at 0.

The functions $(\omega, x) \mapsto h_n(x)1_E(x\omega)$ are Borel, for each ω having at most countable number of x where they do not vanish. By [Lu] Ch. III,IV (cf. Rmk. 2.7.2 below) their sums over x

$$f_n(\omega) := \sum_{x \in G} h_n(x)1_E(x\omega)$$

are Borel functions of ω.

Now apply §A.3. Since each h_n is the increasing limit of non-negative L^1 -functions, there is an embedding of Ω in a G-metrizable compact space s.t. all the convolutions:

$$\tilde{f}_n(\omega) = \int_G f_n(x\omega)h_n(x)\,d\lambda(x)$$

are l.s.c. (lower semi-continuous) functions. It is easy to see that these $\tilde{f}_n$ are the same as f_n, but with U_n^2 instead of U_n.

Now, since $\tilde{f}_n$ are $\{0,\infty\}$-valued l.s.c. functions, the sets $\{\tilde{f}_n = \infty\}$ are open and their intersection E' is G_δ. It is easy to see that this E' will do.

(iii) Metrize G by metrizing the Alexandrov (one-point) compactification of G. Then the Borel structure in the closed sets $\subset G$ is obtained from the Hausdorff metric. Choose a dense countable set $D \subset G$. The mapping which maps each closed $F \subset G$ to the sequence $(\text{distance}(x,F))_{x\in D}$ is 1-1 Borel, hence a Borel isomorphism with the image, thus it is enough to prove $\omega \mapsto \text{distance}(x, \overline{\mathcal{O}_\omega E})$ is Borel for each fixed $x \in D$. But one may write the E' in (ii) as $E' = \cap_{n\geq 1} W_n = \cap_{n\geq 1}\overline{W_n}$, with W_n open and $\overline{W_{n+1}} \subset W_n$ in some relative topology from a G-metrizable compact space in which Ω is embedded. Then

$$\text{distance}(x, \overline{\mathcal{O}_\omega E}) = \sup_n \inf_{y\in D, y\omega\in W_n} \text{distance}(x,y).$$

QED

For the "essential closure" Borel-measurability follows from Fubini.

Thus measurability in ω will follow if we insure that the way the (weighted) hypergraph is constructed from the closed set causes the m-chain made from the (weighted) hypergraph and the simplex of measures to depend measurably on the closed set.

In most of the examples in the sequel, the weighted hypergraph is defined, depending on the closed set, by Borel operations in the points of G and in closed sets (the given closed set, and also, say, a given closed relation, such as a partial ordering). One uses the fact that *in a 2nd-countable locally compact space* finite Boolean operations in closed sets (and also countable intersection) are Borel (a not too hard exercise), and so are $x \mapsto \{x\}$ for $x \in G$ and $(x,F) \mapsto \{y \in G : (x,y) \in F\}$ for $x \in G$, $F \subset G^2$ closed. (Open sets or F_σ sets can also be encoded in a Borel manner using closed sets: encode an open set by its complement, and instead of a variable F_σ set take a variable sequence of closed sets. Yet, one should be cautious: the relation $\cup_{n\geq 1} F_n = F_0$ is co-Suslin, but *not Borel*[6])

Next one applies a simplex of invariant measures, and one has to know that their values on Borel sets, and the integral of Borel functions on them, depend measurably on the sets and functions, in some sense.

A family of $E(\alpha)$ of Borel subsets of a standard Borel space X, depending on a parameter α varying in a standard Borel space $\mathcal{A}$, will be called a **Borel family** if the set

$$\{(x,\alpha) : \omega \in E(\alpha)\} \subset X \times \mathcal{A}$$

is Borel.

Similarly, a family $f(x;\alpha)$ of Borel functions on a standard Borel space X, depending on a parameter α varying in a standard Borel space $\mathcal{A}$, will be called a **Borel family** if $(x,\alpha) \mapsto f(x;\alpha)$ is Borel.

[6]Indeed, the relation $\{x_n\} = F$ for F compact is not Borel in $((x_n)_n, F)$. Otherwise its countable-to-one image: the family of countable compact sets, would be Borel. But for any Borel family $\mathcal{F}$ of countable compact sets, the (countable ordinal) order of the first vanishing derivative is bounded. Indeed (see [Lu] Ch. IV) there is a polish topology in the set of pairs $\{(x,F) : x \in F \in \mathcal{F}\}$ finer than the product of the topologies in $G \times 2^G$. Moreover, we may assume this Polish topology has a countable basis $\mathcal{U}$ composed of clopens. Suppose the order γ of the first vanishing derivatives was unbounded. Then one can find disjoint $U_0, U_1 \in \mathcal{U}$ of diameter $< 2^{-1}$ s.t. $\forall\gamma\exists F \in \mathcal{F}$, $F^{(\gamma)} \cap \{x : (x,F) \in U_j\} \neq \emptyset$ (not having that implies that for γ big enough $F^{(\gamma)}$ is a singleton $\forall F \in \mathcal{F}$). Then one finds disjoint $U_{ij} \in \mathcal{U}$, $i,j \in \{0,1\}$ of diameter $< 2^{-2}$ and s.t. $U_{ij} \subset U_i$ and $\forall\gamma\exists F \in \mathcal{F}$, $F^{(\gamma)} \cap \{x : (x,F) \in U_{ij}\} \neq \emptyset$ and so on. The set $K = \cup U_i \cap \cup U_{ij} \cap \cup U_{ijk} \cap \ldots$ is a Cantor set in the Polish space, and all its elements must be pairs with the same (uncountable) F_∞, otherwise, since the Polish topology is finer than that of $G \times 2^G$, there will be $U_{ij\ldots}$'s without member pairs with common F.

Remark 2.7.2 The counting measure has the following property: for a Borel family of sets (functions), the counting measure of $E(\alpha)$ (the counting integral of $x \mapsto f(x;\alpha)$) is *Borel measurable* in the parameter if all the sets in the family are at most countable (all the functions in the family differ from zero only on a countable set (which depends on the parameter)).

Indeed, By [Lu] Ch. III,IV if X $\mathcal{A}$ are standard and $f : X \times \mathcal{A} \to \overline{\mathbf{R}^+}$ is Borel s.t. for each $\alpha \in \mathcal{A}$, $\omega \in \Omega \mapsto f(\omega, \alpha)$ is different from 0 only on a countable set, then one can write

$$f(x, \alpha) = \sum_{n \geq 1} g_n(\alpha) \delta_{x, \psi_n(\alpha)}$$

for Borel g_n, ψ_n with $g_n \geq 0$. Thus

$$\int_X f(x, \alpha)\, d\mathrm{count}\,_X(x) = \sum_{n \geq 1} g_n(\alpha)$$

is Borel in α.

Definition 2.7.3 *Given a complete measure Λ on a standard Borel space X with all Borel sets measurable. We say that Λ has the* **Borel-** *resp.* **universal- measurability Fubini property** *if for every Borel family $E(\alpha)$ of Borel subsets of X s.t. for each $\alpha \in \mathcal{A}$, $E(\alpha)$ is Λ-σ-finite, the mapping $\alpha \mapsto \Lambda(E(\alpha))$ is Borel-, resp. universally- measurable. Equivalently, if for any Borel family of $\overline{\mathbf{R}^+}$-valued Borel functions on X s.t. for each $\alpha \in \mathcal{A}$, $x \in X \mapsto f(x;\alpha)$ is different from 0 only on a Λ-σ-finite set (which depends on α), the integral $\int_X f(x;\alpha)\, d\Lambda(x)$ is Borel- resp. universally- measurable in α.*

By Fubini any σ-finite completion of a Borel measure has the Borel measurability Fubini property.

By Rmk. 2.7.2, the counting measure has the Borel measurability Fubini property.

Example 2.7.4 An example for a Borel measure which does not have the universal measurability Fubini property: Let $f : [0,1] \to]0, \infty[$ be a function which is not universally measurable and let λ be the Lebesgue measure on $[0,1]$. Define a Borel measure on the square $[0,1]^2$ as follows: a Borel set $E \subset [0,1]^2$ whose projection on the first coordinate is countable has measure $\sum_x f(x) \lambda\{y : (x,y) \in E\}$. If the projection is uncountable the measure is ∞. Then the mass of the member of the Borel family with parameter x – $\{x\} \times [0,1]$, is $f(x)$, which is not universally measurable.

As for other measures Λ on a 2nd-countable locally compact X (such as those mentioned in [Fed] §2.10), one can say the following.

It follows from the fact that for a Borel family of Borel sets, the property ($n \in \mathbf{Z}^+ \cup \{\infty\}$ fixed): "the cardinality of the Borel set $\geq n$" defines a Suslin set of the parameters (it is the set of parameters s.t. the set composed of "sequences of n distinct points in the set" is not empty), and from Fubini, that the universal measurability Fubini property holds for measures whose value on a set E is defined by taking the counting measure of the intersection of E with a variable closed set F and integrating on a fixed (say, invariant) σ-finite measure on F.

This includes k-dimensional measures in $\mathbf{R}^n$, $k < n$ (intersect with $(n-k)$-dimensional affine subspaces and take an invariant Radon measure on these) – see [Fed].

Many measures have the following property: $K \mapsto \Lambda(K)$ is Borel on the set of compact $K \subset X$ (one easily sees that this latter set is Borel in the space of closed sets). We shall say then that Λ is **Borel (measurable) on compacta**.

Indeed, measures whose value on a set E is defined by taking the counting measure of the intersection of E with a variable closed set F and integrating on a fixed (say, invariant) σ-finite measure on F are Borel on compacta – This follows from Borelness of the operations of taking intersection of closed sets and taking the number of elements of a closed set (that number being finite or ∞).

Also, one has Borel measurability on compacta for measures, such as Hausdorff measures, having the following property: there exist a family $(U_{nm})_{n,m \geq 1}$ of open sets in X and numbers $\eta_{nm} \in \overline{\mathbf{R}^+}$ s.t. for compact $K \subset X$:

$$\Lambda(K) = \sup_n \inf_{\{m : K \subset U_{nm}\}} \eta_{nm}$$

(For r-dimensional Hausdorff measure in $\mathbf{R}^N$, let U_{nm} for fixed n enumerate the finite unions of the members of diameter $< 1/n$ in a countable family of open convex sets, such as the family of all finite intersections of rational half-spaces, that have a member between any compact convex set and any of its open neighbourhoods, and take as η_{nm} the infimum of $\sum_{W\in\mathcal{F}}(\text{diameter}W)^r$ over finite coverings $\mathcal{F}$ of U_{nm}, multiplied, if needed, by a normalizing constant.)

Now, if a completion of a Borel measure Λ is Borel on compacta, then it has the universal measurability Fubini property, moreover for a Borel family $\alpha \mapsto E(\alpha) := \{x \in X : (x, \alpha) \in E\}$, $E \subset X \times \mathcal{A}$ Borel, $\mathcal{A}$ standard, s.t. $\forall\alpha$ $E(\alpha)$ is Λ-σ-finite, the set $\{\alpha \in \mathcal{A} : \Lambda(E(\alpha)) > a\}$ is *Suslin*.

Indeed, endow $\mathcal{A}$ with a Polish topology. Then E is a Lusin space hence $\exists$ a Polish space Y and a 1-1 continuous onto $f : Y \to E$. the compositions of f with the projections on X and on $\mathcal{A}$ are continuous. Thus the $\{\alpha\} \times E(\alpha)$ correspond to *closed* subsets $F(\alpha)$ of Y, on which Λ induce σ-finite Borel measures. Since σ-finite Borel measures on Polish spaces are regular (for Borel sets), the masses of the $F(\alpha)$ are the supremum of the masses of compacta $\subset F(\alpha)$.

Therefore $\Lambda(E(\alpha)) > a$ iff $\exists$ a compact $K \subset Y$, mapped onto $\{\alpha\}$ by the projection on $\mathcal{A}$ and mapped by the projection on X onto a (compact) set ($\subset X$) with $\Lambda > a$. Now, the compact sets in a Polish space form, with Hausdorff metric, a Polish space, and mapping compact sets to their image via a continuous function is continuous. Thus it follows from Λ being Borel on compacta in X that $\{\alpha : \Lambda(E(\alpha)) > a\}$ is Suslin.

Similar considerations show that the *product* of two measures Λ_1 and Λ_2 on two 2nd-countable locally compact X_1 and X_2, both completions of Borel measures, has the universal measurability Fubini property, if it is assumed that both measures are Borel on compacta, and the product is defined so that only sets contained in a σ-finite Borel "rectangle" in $X_1 \times X_2$ can have finite product mass.

Indeed, if both measures are Borel on compacta, then the product measure is Borel on compact sets $K \subset X_1 \times X_2$ with projections on X_1 and X_2 having finite mass (The $\Lambda_1 \otimes \Lambda_2$-mass of such K is equal to an integral $\int_{K_1(K)} f(K, x_1)\, d\Lambda_1(x_1)$ with a Borel function $f(K, x_1)$ and with $K_1(K)$ a compact in X_1 of finite Λ_1-mass which depends in a Borel manner on K. To prove such an integral is Borel in K, one may assume the Borel function f is the characteristic function of a Borel set F of (K, x_1)'s, and since the collection of F's satisfying what we want is *monotone* ([Ha-M] §6), i.e. stable w.r.t. unions and intersections of monotone sequences of Borel sets, and contains the Boolean algebra consisting of the finite disjoint unions of "rectangles" with one side a Borel set of K's and the other side a difference of closed sets in X_1 – here we use the fact that intersecting a fixed closed set in X_1 with $K_1(K)$ is a Borel operation in K_1 – it contains all Borel sets.) Then to prove the universal measurability Fubini property of the product measure we proceed as above, restricting the compact sets in the Polish space Y to those having projections on X_1 and X_2 with finite mass.

Remark 2.7.5 Note that for a Borel action of a 2nd-countable locally compact group G on a standard Borel space Ω, *satur* E is, for any Borel $E \subset \Omega$, a projection of a Borel set in a product, hence Suslin, thus universally measurable. (One cannot say more in general: for a projection $\mathrm{pr}_1(E)$ of a Borel set $E \subset \mathbf{R}^2$ on $\mathbf{R}^1$ (which need not be Borel), we have $\mathrm{pr}_1(E) \times \mathbf{R} = \mathit{satur}\,E$ for the action of $\mathbf{R}$ on $\mathbf{R}^2$ via shift on one coordinate.)

If E has the property that $\mathcal{O}_\omega E$ is *countable* for all $\omega \in \Omega$, then *satur* E is Borel since the image of a countable-to-one Borel mapping among standard spaces is Borel ([Lu] Ch. III,IV – see the beginning of this §).

Even when *satur* E is universally measurable but not Borel, one can find, for every invariant Borel probability measure μ in Ω, an invariant Borel set E', differing from *satur* E in a (μ-)null set and containing it, resp. is contained in it (in which case $E' = \mathit{satur}(E \cap E')$, with $E \cap E'$ differing from E in a $\frac{d\Lambda}{d\lambda}\mu$-null set for any Λ) as is shown by the following

Proposition 2.7.6 *Let a 2nd-countable locally compact group G act in a measure-preserving Borel manner on a σ-finite standard measure space $(\Omega, \mathcal{B}, \mu)$.*

(i) Let $f : \Omega \to \overline{\mathbf{R}^+}$ be Borel and almost-invariant in the sense that for each fixed $x \in G$ $f(x\omega) = f(\omega)$ a.e. Then $\exists$ an invariant Borel function $\tilde{f} : \Omega \to \overline{\mathbf{R}^+}$ s.t. $\tilde{f}(\omega) = f(\omega)$ a.e. and s.t. if ω is s.t. $\mathcal{O}_\omega f$ is constant a.e. (w.r.t. Haar measure on G) then $\tilde{f}(\omega)$ is equal to this constant. Moreover, $\tilde{f}$ can be chosen s.t. there is an embedding of Ω as an invariant Borel subset of a G-metrizable compact space s.t. $\tilde{f}$ is l.s.c. (lower semi-continuous) for the relative topology on Ω.

(ii) *Let $E \subset \Omega$ be Borel and almost-invariant (i.e. $\forall$ fixed $x \in G$ $E \triangle xE$ is null). Then $\exists$ an invariant Borel set $\tilde{E}$ which differs from E only on a null set, and s.t. if ω is s.t. $\mathcal{O}_\omega E$ is null (resp. conull) w.r.t. Haar measure on G, then $\omega \notin \tilde{E}$ (resp. $\omega \in \tilde{E}$).*

(iii) *Every measurable invariant set E differs by a null set from some Borel invariant set $\tilde{E}$ contained in it (resp. which contains it).*

Proof (i) We refer to §A.3.

Let λ be a Haar measure on G and let h be some non-negative L^1 function on G. Let f' be the Borel $function$ on Ω:

$$f'(\omega) := \int_G f(x\omega)h(x)\,d\lambda(x).$$

The functions $(\omega, x) \mapsto f(x\omega)$ and $(\omega, x) \mapsto f(\omega)$ on $\Omega \times G$ are Borel, and for every fixed x are μ-a.e. equal. Therefore they are $\mu \otimes \lambda$- a.e. equal (here we use the fact that μ is σ-finite!), hence for μ-a.a. ω $\mathcal{O}_\omega f$ is equal λ-a.e. to the constant $f(\omega)$, and for these ω's $\mathcal{O}_\omega f'$ is the constant $f(\omega)$.

By §A.3 there is an embedding of Ω as an invariant Borel subset of a G-metrizable compact space K s.t. f' can be extended to a l.s.c. function on K. Let the function $\tilde{f}$ on Ω be:

$$\tilde{f}(\omega) := \sup_{x \in G} f'(x\omega)$$

For each ω s.t. $\mathcal{O}_\omega f$ is λ-a.e. the constant $f(\omega)$, we have $\tilde{f}(\omega) = f(\omega)$. Also $\tilde{f}$ is invariant and l.s.c. Thus it satisfies our requirements.

(ii) follows from (i).

(iii) Choose a Borel set $E' \subset E$ (resp. $E' \supset E$) which differs from E by a null set. For every $x \in G$ $xE = E$ and xE' and E' differ from it by null sets. Therefore E' is almost-invariant. An invariant Borel set $\tilde{E}'$ as in (ii) will do.

QED

2.8 An Assortment of Kac-like Theorems, Continuous Case

CHG gives us analogs of the Kac-like theorems of Section 1, *mutatis mutandis.*

Recall that, when we deal with a (weighted) *graph* and a 1-simplex of invariant measures, *the source is the target measure of the set of targets of arrows with source* 0, analogously for the target.

2.8.1 A Continuous Kac Formula

Let us start with the promised (see Example 2.4.4, cf. [Bi])

Theorem 2.8.1 A Continuous Kac Theorem *Let* $\mathbf{R}$ *act measure-preservingly and in a Borel manner on a standard probability measure space* (Ω, μ)*. Let* $E \subset \Omega$ *be Borel.*

Define:

$$\begin{aligned} ex\,E &:= \{\omega \in \Omega : \forall \varepsilon > 0 \ \omega \text{ visits } E \text{ in the }]0,\varepsilon[\text{-past}\} \\ in\,E &:= \{\omega \in \Omega : \forall \varepsilon > 0 \ \omega \text{ visits } E \text{ in the }]0,\varepsilon[\text{-future}\} \end{aligned}$$

(with obvious meaning for "visiting" etc.)

Note that $E \cup ex\,E \cup in\,E = \{\omega \in \Omega : 0 \in \overline{\mathcal{O}_\omega E}\}$.

Define the **return time** $\rho(\omega) = \rho_E(\omega) : \Omega \to [0, \infty]$ *by:*

$$\rho_E(\omega) := \begin{cases} \inf\{t > 0; T^t\omega \in E\} & \text{if } \omega \in E \cup ex\,E \\ 0 & \text{otherwise} \end{cases}$$

Note that on $(E \cup ex\,E) \setminus in\,E$ *we have* $\rho > 0$.

Then $in\,E$, $ex\,E$, $E \cup ex\,E$ *and* $E \cup in\,E$ *differ only in null sets and if* $\tilde{E}$ *is any set differing from them by a null set (such as* $E \cup in\,E \cup ex\,E$*) then we have:*

$$\mathbf{E}\left(\rho \frac{dcount}{dt}\right) = \mu((satur\,E) \setminus \tilde{E})$$

in other words,

$$\mathbf{E}\left(\rho \frac{dcount}{dt} + 1_{\tilde{E}}\right) = \mu(satur\,E) \tag{24}$$

Note that if a.e. $\mathcal{O}_\omega E$ *is* closed, *one may take* $\tilde{E} = E$*; if a.e.* $\mathcal{O}_\omega E$ *is* discrete, *one may take* $\tilde{E} = \emptyset$.

Proof (note the analogy with our proof of the usual Kac in §1.1)

First, we have Poincaré recurrence in the following sense: for a.a. $\omega \in satur\,E$ E is visited in any $[t_0, \infty[$-past or future. Indeed, the sets

$$A_n = \{\omega : [n, n+1[\text{ is the last such interval intersecting } \mathcal{O}_\omega E\}$$

are in the Boolean algebra generated by Suslin sets, hence are measurable, are disjoint and have the same measure.

Now apply CHG to the ω-dependent graph consisting of the arrows (t_0, t_1) s.t. $t_0 > t_1$ and $[t_1, t_0] \cap \overline{\mathcal{O}_\omega E} = \{t_1\}$, and for the 1-simplex of invariant measures on $\mathbf{R}$ $(dt, \text{count}_{\mathbf{R}})$. (Check that the conditions for applying CHG are satisfied.)

The *source* equals a.e. $1_{satur\,E \setminus (E \cup ex\,E \cup in\,E)}$.

The *target* equals ρ_E.

which, by CHG, proves the asserted formula for $\tilde{E} = E \cup ex\,E \cup in\,E$. This shows that ρ_E has finite $\frac{d\text{count}}{dt}\mu$-integral, hence, by Prop. 2.4.6, has zero μ-integral, hence $\mu((E \cup ex\,E) \setminus in\,E) = 0$. Applying this for the inverse action one deduces $\mu((E \cup in\,E) \setminus ex\,E) = 0$, and we are done.

QED

The second term in the left-hand side of (24) may be interpreted as the result of points ω in $in\,E$ having "infinitesimal return time" (indeed, "having a.e. return time $\frac{dt}{d\text{count}}$" !?).

Remark 2.8.2 Here one may appreciate the need for the $\prod \Lambda_i$-σ-finiteness assumption in the formulation of CHG. Let us change the graph in the proof of 2.8.1 by requiring $t_0 \geq t_1$ instead of $t_0 > t_1$, thus adding all the arrows (t, t), $t \in \overline{\mathcal{O}_\omega E}$ to the graph. This does not change the target, but the source is now a.e. $1_{satur\,E}$ and the formula that might be obtained is incorrect. Indeed, the σ-finiteness is not satisfied (for $\text{count}_{\mathbf{R}}$).

Remark 2.8.3 Let $T > 0$, and let us integrate the test-function (nonegative Borel) $1_{[0,T[}$ on the 0-chain corresponding to the enhanced function under the expectation in (24), for $\tilde{E} = E \cup in\,E \cup ex\,E$, thus $\mathcal{O}_\omega 1_{\tilde{E}} = 1_{\overline{\mathcal{O}_\omega E}}$. The integral on $1_{\overline{\mathcal{O}_\omega E}}\,dt$ is complemented by the integral on $\mathcal{O}_\omega \rho\, d\text{count}_{\mathbf{R}}$ and we have the following fact, equivalent, in fact, to (24):

$$\forall T > 0 \quad \frac{1}{T} \int_\Omega [\inf(\mathcal{O}_\omega E \cap]T, \infty[\,) - \inf(\mathcal{O}_\omega E \cap]0, \infty[\,)]\, d\mu(\omega) = 1 \tag{25}$$

(Taking, e.g., $[0, \infty]$ instead of $]0, \infty]$ will change the integrand only on $E \setminus in\,E$, a null set by Thm. 2.8.1.)

The integrand is 0 if ω will not enter E in the $[0, t]$-future, otherwise it is the time from its "first" future visit to E to its "first" one after time T.

One may define the **arrival time** to E $\xi(\omega) = \xi_E(\omega)$ by

$$\xi_E(\omega) := \inf \mathcal{O}_\omega E \cap]0, \infty[$$

(One might take $[0, \infty[$ instead of $]0, \infty[$, changing ξ only on a null set.)

As in the discrete case (Rmk. 1.2.5), If we knew that ξ_E is L^1 in ω, we could prove (25) in one line.

For the rest of this section, unless stated otherwise, the setting is that of Borel measure-preserving action on a standard Borel probability space $(\Omega, \mathcal{B}, \mu)$, and E borel.

2.8.2 Some Assertions with No Discrete Analog

Note first, that the mere defining property (19) of the expectation of an enhanced function, which can be written as:

$$\mathbf{E}\left(\int_G h(x)f(x\omega)\,d\Lambda(x)\right) = \mathbf{E}f(\omega)\frac{d\Lambda}{d\lambda}\int_G h(x)\,d\lambda(x) \tag{26}$$

is a case of CHG with weighted graph $F(x,y;\omega) = h(xy^{-1})f(x\omega)$ and 1-simplex of measures (Λ, λ).

An application which involves two invariant measures, thus has no discrete analog, comes from the weighted graph $f(x\omega)h(xy^{-1})$ and 1-simplex of measures $(\mathrm{count}_G, \Lambda)$, where $f \geq 0$ is $\frac{d\mathrm{count}}{d\lambda}\mu$-integrable (really it is enough that f is Borel and $\mathcal{O}_\omega f$ is a.e. supported on a countable set) and h is non-negative s.t. $h(x^{-1})$ is Λ-integrable. Note that the σ-finiteness requirement of CHG is satisfied – the set of relevant y's for each fixed ω is Λ-σ-finite. To check measurability of the obtained 1-chain we may view x and yx^{-1} as two independent variables. This separates f and h. (Note that yx^{-1} is restricted to a fixed Λ-σ-finite set.) It suffices to check test-functions on G^2 where these variables are similarly separated, i.e. of the form $g_1(x)g_2(yx^{-1})$. For the variable x we may use Rmk. 2.7.2. We conclude that the obtained 1-chain is measurable, and CHG gives the following:

$$\text{for } g(\omega) := \int_G f(x\omega)h(x)\,d\mathrm{count}_G, \quad \mathbf{E}g(\omega)\frac{d\Lambda}{d\lambda} = \left[\int_G h(x^{-1})\,d\Lambda(x)\right]\mathbf{E}f(\omega)\frac{d\mathrm{count}}{d\lambda} \tag{27}$$

For example, referring to Exm. 2.4.7 item 2, if $S \subset \mathbf{R}$ has finite measure w.r.t. some Hausdorff measure Λ on $\mathbf{R}$ then the union of the translates of a smooth curve by S (with multiplicity) has finite $\frac{d\Lambda}{dt}\mu$-mass.

The formulas in this § have applications to the problem of recovering the original measure from the infinitesimal measure (§4.3).

2.8.3 Some Analogs to the Discrete Case

One may formulate analogs to the facts in §1.2, the analogy being both in formulation and in proof.

Note that assertions like the continuous Kac Thm. 2.8.1 are more lucid in the case when for a.a. $\omega \in \Omega$ $\mathcal{O}_\omega E$ is *closed*, and in most of what follows it will be assumed that $\mathcal{O}_\omega E$ is *discrete* in $\mathbf{R}$. Such is the case for Exm. 2.4.7 item 6 (Poisson distribution), or for other stationary processes defined by probability over the discrete subsets of G. For $\mathbf{R}$-action, we have by Poincaré's recurrence that when $\mathcal{O}_\omega E$ is a.e. discrete, it is unbounded above and below a.e. in *satur* E.

Note that if $\mathcal{O}_\omega E$ is a.e. discrete then E is always μ-null (Fubini for μ and Haar in G). Of course, it may be non-null w.r.t. infinitesimal measures.

Proposition 2.8.4 *Let $\mathbf{R}$ act in a measure-preserving Borel manner on a standard probability space (Ω, μ). Suppose $E \subset \Omega$ is s.t. $\forall\omega \in \Omega$ $\mathcal{O}_\omega E$ is* discrete *in $\mathbf{R}$. Then the $\mathbf{Z}$-action on $T_E : E \to E$ given by the* **induced transformation** *(see §1.2.1):*

$$T_E(\omega) := T^{\min(\mathcal{O}_\omega E \cap]0,\infty])}\omega \quad \omega \in E$$

is Borel, and, w.r.t. $\frac{d\mathrm{count}}{dt}\mu$ on E, defined a.e. and measure preserving.

Proof First, let Ω_1 be the set

$$\{\omega \in \Omega : \mathcal{O}_\omega E \text{ is unbounded above and below.}\}$$

Ω_1 is invariant, it is Borel since the discrete set $\mathcal{O}_\omega E$ depends in a Borel manner on ω (Prop. 2.7.1), and by Poincaré recurrence it is conull. Therefore, by the definition of $\frac{d\mathrm{count}}{dt}\mu$, $E \cap \Omega_1$ is $\frac{d\mathrm{count}}{dt}\mu$-conull in E, and we may and do replace Ω by Ω_1 and E by $E \cap \Omega_1$, making T_E 1-1, onto and Borel on E.

Let $f : E \to \mathbf{R}^+$ be Borel $\frac{d\mathrm{count}}{dt}\mu$-integrable.

Weighted graph: the arrows (t_0, t_1) s.t. $t_1 < t_0$ and $[t_1, t_0] \cap \mathcal{O}_\omega E = \{t_1, t_0\}$, with weight $f(T^{t_0}\omega)$.

1-simplex of measures: $(\mathrm{count}_{\mathbf{R}}, \mathrm{count}_{\mathbf{R}})$.

Check that CHG is applicable. Note that the weighted graph is a Borel function of (the discrete $\mathcal{O}_\omega E, t_0, t_1$).

We have: source: $f(\omega)$; target: $f(T_E(\omega))$.

CHG implies $f(T_E(\omega))$ is $\frac{d\mathrm{count}}{dt}\mu$-integrable with the same integral as $f(\omega)$. Consequently, T_E is measure-preserving.

QED

In order to show what happens, in the continuous case, to assertions such as Propositions 1.2.2, 1.2.1 and 1.2.3, let us state:

Proposition 2.8.5 *For measure-preserving Borel* $\mathbf{R}$*-action on a standard* (Ω, μ)*, let* $E \subset \Omega$ *be Borel and assume* $\mathit{satur}\, E = \Omega$ *and* $\mathcal{O}_\omega E$ *always discrete. Define the return time* $\rho = \rho_E$ *and the arrival time* $\xi = \xi_E$ *as usual (*ρ_E *is* 0 *outside* E*). Let the superscript* $(-)$ *refer to the inverse action. Then*

- ξ_E *and* $\xi_E^{(-)}$ *have the same ordinary (i.e.* μ*-) distribution.*
- ρ_E *and* $\rho_E^{(-)}$ *have the same* $\frac{dcount}{dt}\mu$*-distribution.*
- *For any* $a \geq 0$*, the sets* $\{\xi_E = a\}$ *and* $\{\xi_E^{(-)} = a\}$ *have the same* $\frac{dcount}{dt}\mu$*-mass.*
- *For any* $a \geq 0$*, the* $\frac{dcount}{dt}\mu$*-masses of* $\{\xi_E = a\}$ *and* $\{\rho_E > a\}$ *are equal.*

Now let $s : \mathbf{R}^+ \to \mathbf{R}^+$ *be Borel and let* $S(x) := \int_0^x s(t)\, dt$*. We have:*

- $$\mathbf{E}s(\xi_E(\omega)) = \mathbf{E}S(\rho_E(\omega))\frac{dcount}{dt}$$

  ($s \equiv 1$ *gives Kac's.)*

(Note that the last assertion is not *a direct consequence of the preceding one – see Exm. 2.4.7 item 7.)*

Proof Exercise in using CHG, in analogy with §1.2.1.

QED

Thus, ξ is in L^p iff ρ is in L^{p+1} for $\frac{d\mathrm{count}}{dt}\mu$.

In the two last assertions of the above Cor. one could, in retrospect, integrate the infinitesimal measure of $\{\chi = a\}$ as in Exm. 2.4.7 item 7. One may try to formulate general conditions for the possibility of such integration for a stochastic variable.

As an exercise, one may formulate and prove assertions about *return and arrival to two sets*, in analogy with Prop. 1.2.7.

Now, for $\mathbf{R}$-action with discrete $\mathcal{O}_\omega E$, an analog of Thm. 1.2.8 about repeated return and arrival can be formulated, referring to ordinary (μ-) and $\frac{d\mathrm{count}}{dt}\mu$- distributions. We mention just the analog of (13) (see Rmk. 1.2.9) which will give us an opportunity to apply CHG to a weighted 2-hypergraph, composed of 2-simplices.

Assume measure preserving Borel $\mathbf{R}$-action on a standard (Ω, μ), $E \subset \Omega$ Borel with $\mathit{satur}\, E = \Omega$ and $\mathcal{O}_\omega E$ always discrete.

Let $s, s' : \mathbf{R}^+ \to \overline{\mathbf{R}^+}$ be Borel, and consider their *convolution*: $(s * s')(x) = \int_0^x s(t)s'(x-t)\, dt$.

The weighted 2-hypergraph:

$$\{(t_0, t_1, t_2) : t_1 < t_0 < t_2,\ [t_1, t_2] \cap \overline{\mathcal{O}_\omega E} = \{t_1, t_2\}\}$$

with weight $s(t_2 - t_0)s'(t_0 - t_1)$.

The 2-simplex of measures: $(dt, \mathrm{count}_{\mathbf{R}}, \mathrm{count}_{\mathbf{R}})$.

Check that CHG is applicable. The three vertices are:

the 0-vertex: $s(\xi_E(\omega))s'(\xi_E^{(-)}(\omega))$
the 1-vertex: $(s * s')(\rho_E(\omega))$
the 2-vertex: $(s * s')(\rho_E^{(-)}(\omega))$

and one gets the following continuous variant of (13) – a "decomposition" of Kac's formula:

$$\mathbf{E}\left[s(\xi(\omega))s'(\xi^{(-)}(\omega))\right] = \mathbf{E}\left[(s * s')(\rho(\omega))\frac{d\text{count}}{dt}\right] \tag{28}$$

Kac's obtains from $s = s' = 1_{]0,\infty[}$; then $(s * s')(x) = x$.

Let us conclude with further analogs of §1.2.1. Retain the setting of $\mathbf{R}$-action and $E \subset \Omega$ Borel with $satur\, E = \Omega$ and $\mathcal{O}_\omega E$ always discrete. Let $s : \mathbf{R}^+ \to \overline{\mathbf{R}^+}$ be Borel and let $S(x) := \int_0^x s(t)\, dt$.

Consider our usual Kac graph:

$$\{(t_0, t_1) : t_0 > t_1,\ [t_1, t_0] \cap \overline{\mathcal{O}_\omega E} = \{t_1\}\}$$

and the 1 simplex of measures $(dt, \text{count}_{\mathbf{R}})$.

But now we turn the graph into a weighted graph, giving weights to its 1-simplices, in two different manners:

Let $f : \Omega \to \mathbf{R}^+$ be Borel and consider the weight $f(T^{t_0}\omega)s(t_0 - t_1)$, CHG gives:

$$\int_\Omega f(\omega)s(\xi^{(-)}\omega)\, d\mu(\omega) = \int_E \left(\int_0^{\rho(\omega)} f(T^t\omega)s(t)\, dt\right) \frac{d\text{count}}{dt}\, d\mu(\omega) \tag{29}$$

Now let $f : E \to \mathbf{R}^+$ be Borel and consider the weight $f(T^{t_1}\omega)s(t_0 - t_1)$. CHG gives:

$$\int_\Omega f\left(T^{-\xi^{(-)}\omega}(\omega)\right) s(\xi^{(-)}\omega)\, d\mu(\omega) = \int_E f(\omega)S(\rho(\omega))\, \frac{d\text{count}}{dt}\, d\mu(\omega) \tag{30}$$

Note that, in fact, (29) for $s \equiv 1$, with suitable f's, encompasses both (29) and (30).

Remark 2.8.6 (compare with Rmk. 1.2.6). Ambrose and Kakutani [AK] (see also [J], [Na]) have shown, for measurable $\mathbf{R}$-action, that, taking apart an invariant subset where the action is trivial on the Boolean Algebra of measurable sets modulo null sets, there always exists an E s.t. $\mathcal{O}_\omega E$ is discrete a.e. and $satur\, E$ is conull. Then Ω has the structure of the "flow under a function" construction (see [Pe], p. 11), while (29) for $s \equiv 1$ shows that μ is indeed the measure making $(\omega, \mu, (T^t)_{t\in\mathbf{R}})$ the result of the "flow under a function" construction starting from the discrete system $(E, \frac{d\text{count}}{dt}\mu|_E, T_E)$ and the function $\rho_E : E \to [0, \infty[$. Note that by Kac's ρ_E has integral 1 on $(E, \frac{d\text{count}}{dt}\mu|_E)$.

Thus, in this case Ω can be recovered from E and μ can be recovered from the infinitesimal measure $\frac{d\text{count}}{dt}\mu$. More on this in §4.3.

One can play with CHG to find analogs, for continuous preordered Time, (take $G = \mathbf{R}^n$), to what is said in §1.2.3. In particular, if $\mathcal{O}_\omega E$ is always closed, one can define the return and arrival duration and epoch, and one has formulas analogous to §1.2.3,

As an example of such statement, one has the following, which for $n = 1$ and the usual ordering gives continuous Kac's:

Proposition 2.8.7 *Let $\mathbf{R}^n$ act in a Borel and measure-presrving manner on a standard probability space $(\Omega, \mathcal{B}, \mu)$.*

Fix a preordering $\geq$ in $\mathbf{R}^n$ with F_σ graph.

Let $E \subset \Omega$ be Borel with $\mathcal{O}_\omega E$ always closed. *(W.l.o.g. one may think of $\Omega =$ the Borel space of closed subsets $\subset \mathbf{R}^n$ with some shift-invariant probability measure and $E = \{\omega : 0 \in \omega\}$.)*

Let Λ be some invariant measure in $\mathbf{R}^n$ which is a completion of a Borel measure and has the universal measurability Fubini property (Def. 2.7.3).

Denote the Lebesgue measure in $\mathbf{R}^n$ by dx.

Then:

- *The expectation of the Lebesgue mass of the arrival duration:*

$$\{x \in \mathbf{R}^n : [0,x] \cap \mathcal{O}_\omega E = \emptyset\}$$

is equal to the expectation of the Lebesgue mass of the arrival duration w.r.t. the inverse action.

Also, for any $z \in \mathbf{R}^n$*, the* $\frac{d\,count}{dx}\mu$*-mass of the set of* ω *s.t.* z *is in the arrival duration is equal to the same for the inverse action.*

- *Suppose that for all* ω *the set*

$$\{y \in \mathcal{O}_\omega E : \exists x \in \mathbf{R}^n\ y < x,\ [y,x] \cap \mathcal{O}_\omega E = \{y\}\}$$

($x > y$ *means* $x \geq y \,\&\, x \neq y$*)*

is Λ*-*σ*-finite.*

Then the $\frac{d\Lambda}{dx}\mu$*-integral over* E *of the Lebesgue mass of the return duration:*

$$\{x \in \mathbf{R}^n : x > 0,\ [0,x] \cap \mathcal{O}_\omega E = \{0\}\}$$

is equal to the expectation of the Λ*-mass of the arrival epoch w.r.t. the inverse action:*

$$\{x \in \mathbf{R}^n : x > 0,\ [-x,0] \cap \mathcal{O}_\omega E = \{-x\}\}$$

QED

Here one may think of E s.t. $\mathcal{O}_\omega E$ is discrete, with the $\Lambda = \mathrm{count}$. For a non-discrete case take Exm. 2.4.7 item 6 – the case of Brownian motion in R^n or the case of Poisson's distribution, but $E =$ the ω's s.t. 0 is distanced $< r_0$ from a point in the discrete set ω. Here invariant measures different from count come into play. As another example check the above for the continuous (and simpler) analog of Exm. 1.2.14 (triangle in the torus $\mathbf{T}^2$ acted by $\mathbf{R}^2$).

2.8.4 The Nearest Point

This construction gives some multi-dimensional continuous analog to the one-dimensional "future until the first return" engaged in Kac's and in the "flow under a function" construction. It will be applied in §4.3.3.

Let G be a *(connected) unimodular Lie group* acting in a Borel and measure preserving manner on a standard probability (Ω, μ). Choose a positive definite symmetric bilinear form on the tangent space T_eG and by right-shifting it to each tangent space T_xG via the derivative at e of $y \mapsto yx$, construct a Riemannian metric, turning G into a Riemannian manifold with right-invariant metric. Note that since the exponential map at e defined by geodesics (see [Hi]) is the same as the Lie-group-theoretic exponential map, the exponential map is everywhere defined, hence G is complete as a Riemannian manifold.

Let $E \subset \Omega$ and assume for simplicity that $\mathcal{O}_\omega E$ is always *closed*.

What we have in mind is to take the graph

$$\{(x,y) \in G^2 : y \text{ is the unique nearest point to } x \text{ in } \mathcal{O}_\omega E.\} \tag{31}$$

To this end, the following proposition is helpful:

Proposition 2.8.8 *Let* X *be a complete Riemannian manifold. Let* $Y \subset X$ *be closed. Then if* $x_0 \in X$ *is a point where* $x \mapsto distance(x,Y)$ *(the distance from* x *to* Y*) is differentiable giving gradient* $-v \in T_xX$ *(that is, it has a differential equal to the linear functional* $w \mapsto \langle -v, w\rangle$ *where* $\langle , \rangle$ *is the inner product), then there is a unique point in* Y *nearest to* x*, that point being*

$$\exp_{x_0}(distance(x_0,Y)v)$$

(that is, the point with parameter 1 *on the geodesic emanating from* x *with tangent vector* $distance(x_0,Y)v$ *– see [Hi]).*

Note that since $x \mapsto distance(x,Y)$ *is Lipschitz, it is differentiable a.e. by Rademacher's Theorem (see [Fed] §3.1.6.). (Note that in the neighbourhood of each point the Riemannian metric is Lipschitz-equivalent to the Euclidean metric of a coordinate chart.) Thus for a.a.* x *there is a unique nearest point.*

Proof First, in a complete Riemannian manifold any bounded closed set is compact, therefore there are nearest points y, and it suffices to prove that if there is a differential giving gradient $-v$ then each nearest point y equals $\exp_{x_0}(\text{distance}(x_0, Y)v)$.

Let $t \mapsto \exp_{x_0} tv'$ be the shortest geodesic from x_0 to y (this exists in a complete Riemannian manifold), with $\exp_{x_0} v' = y$. (thus $\|v'\| = \text{distance}(x_0, Y)$). Clearly,

$$\text{distance}\left(\exp_{x_0} tv', y\right) = (1-t)\|v'\| \quad 0 \le t \le 1,$$

and by computing the derivative at x_0 in this direction we find

$$\langle -v, v'\rangle = -\|v'\|.$$

But $\|v\| \le 1$, since the Lipschitz constant of $x \mapsto \text{distance}(x, Y)$ is ≤ 1. This implies $v' = \|v'\|v$, and $y = \exp_{x_0}(\text{distance}(x_0, Y)v)$.

QED

Proposition 2.8.9 *In the above situation, the set of (x, Y), $x \in X$, $Y \subset X$ closed, satisfying: there is a unique nearest point to x in Y, is closed in the product $X \times 2^X$ (2^X denotes the Borel space of closed subsets of X – see §2.7), and the mapping which maps each (x, Y) in this set to the unique nearest point is Borel.*

Proof Let D be a fixed dense set in X. There is a unique nearest point iff for each $n \in \mathbf{N}$ there are rational $q_1 < q_2$ and $x \in D$ with $q_2 - q_1 < 1/n$, the closed ball $B(x, q_1)$ not intersecting Y and the closed ball $B(x, q_2)$ intersecting Y in a non-empty set $\subset B(x, 1/n)$. y is the unique nearest point if one may require also $y \in B(x, 1/n)\ \forall n$. Since a closed ball is a continuous function of its center and radius (Hausdorff topology in the space of compact sets), and intersection of closed sets is a Borel function, we are done.

QED

Return now to our connected unimodular Lie group G acting on Ω (choose a Haar measure λ on G) and to our $E \subset \Omega$, and assume $\mathcal{O}_\omega E$ is always *countable closed.* For $\omega \in \Omega$, denote by $\pi(\omega) = \pi_E(\omega) \in G$ the unique nearest point to 0 in $\mathcal{O}_\omega E$, if it exists (If there is no unique nearest point, $\pi_E(\omega)$ is undefined); for $\omega \in E$, denote by $P(\omega) = P_E(\omega)$ the set of $x \in G$ s.t. 0 is the unique nearest point to x in $\mathcal{O}_\omega E$.

Taking into account Prop. 2.7.1 and Prop. 2.8.9 above, we have that the set Ω' where π_E is defined is Borel and $\pi_E : \Omega' \to G$ is Borel, moreover $\mathcal{O}_\omega \Omega'$ is conull (w.r.t. Haar) $\forall \omega \in$ *satur* E. Also $\{(\omega, x) : x \in P_E(\omega) \subset \Omega \times G\}$ is Borel.

So consider the ω-dependent graph (31). Since the Riemannian metric is right-invariant, so is the graph, and it will remain so if we take a weight $f(x\omega)s(xy^{-1})$ or $f(y\omega)s(xy^{-1})$, s, f Borel. Take the 1-simplex of measures $(\lambda, \text{count}_G)$. The σ-finiteness requirements of CHG are satisfied. By the above, the relation: "(x, y) belongs to the graph at ω" is Borel in (x, y, ω). Also $\mathcal{O}_\omega E$ is always countable. Therefore (see Rmk. 2.7.2) the obtained 1-chain is Borel-measurable. By Prop. 2.8.8, the source of the unweighted graph is a.e. $1_{satur\,E}$. CHG gives now the following statements, in the spirit of Kac's (compare (29) and (30)):

Proposition 2.8.10 *Let G be a connected unimodular Lie group acting in a Borel and measure-preserving manner on a standard probability space (Ω, μ). Let $E \subset \Omega$ be Borel s.t. $\mathcal{O}_\omega E$ is always non-empty countable closed. (Thus satur $E = \Omega$.) Endow G with a right-invariant Riemannian structure and let λ be a Haar measure in G.*

For $\omega \in \Omega$, let $\pi_E(\omega) \in G$ be the unique nearest point (w.r.t. the Riemannian metric) to 0 in $\mathcal{O}_\omega E$, if it exists. If there is no unique nearest point, $\pi_E(\omega)$ is undefined.

For $\omega \in E$, let $P(\omega)$ be the set of $x \in G$ s.t. 0 is the unique nearest point (w.r.t. the Riemannian metric) to x in $\mathcal{O}_\omega E$.

Then ((i) is, of course, a special case of (ii) which is a special case of (iii) or (iv), these being special cases of (iii) for $s \equiv 1$):

(i) (consider the above unweighted graph (31). The simplex of measures is always $(\lambda, count_G)$)

$$\int_E \lambda(P(\omega))\, \frac{d count}{d\lambda}\, d\mu(\omega) = 1$$

(ii) (consider the above graph (31 with weight $s(xy^{-1})$)

$$\int_\Omega s\left((\pi_E\omega)^{-1}\right)\,d\mu(\omega) = \int_E \left(\int_{P(\omega)} s(x)\,d\lambda(x)\right)\frac{dcount}{d\lambda}\,d\mu(\omega) \tag{32}$$

(iii) Let $f : \Omega \to \overline{\mathbf{R}^+}$ and $s : G \to \overline{\mathbf{R}^+}$ be Borel. Then (consider the above graph (31) with weight $f(x\omega)s(xy^{-1})$):

$$\int_\Omega f(\omega)s\left((\pi_E\omega)^{-1}\right)\,d\mu(\omega) = \int_E \left(\int_{P(\omega)} f(x\omega)s(x)\,d\lambda(x)\right)\frac{dcount}{d\lambda}\,d\mu(\omega) \tag{33}$$

(iv) Now let $f : E \to \overline{\mathbf{R}^+}$ and $s : G \to \overline{\mathbf{R}^+}$ be Borel. Then (consider the above graph (31) with weight $f(y\omega)s(xy^{-1})$):

$$\int_\Omega f\left(\pi_E\omega\right)s\left((\pi_E\omega)^{-1}\right)\,d\mu(\omega) = \int_E f(\omega)\left(\int_{P(\omega)} s(x)\,d\lambda(x)\right)\frac{dcount}{d\lambda}\,d\mu(\omega) \tag{34}$$

3 Equidecomposability and the Totality of Invariant Measures

3.1 Introduction

In this section the acting group G is assumed *discrete.*

As we have seen, Kac's theorem is a special case of equidecomposable functions which trivially have the same integral with respect to the G-invariant measure.

A significant fact, however, is that the property of functions to be equidecomposable (such as $\rho_E(\omega)$ and $1_{satur'E}$ in the formulation of Kac's in §1.1) has nothing to do with the particular invariant measure. This suggests an investigation of the relationship between equidecomposability and *the totality of invariant measures* in suitable frameworks.

The results which will be presented try to state reverse implications: if two functions have the same integral (or one has always a greater integral) w.r.t. a comprehensive set of invariant measures, then they are close to being equidecomposable in a suitable sense (or one may find a function equidecomposable with one function and dominated by the other, etc.).

As an example to such "reverse implication", one deduces directly from the Hahn-Banach Thm. that for a group acting by homeomorphisms on a compact space, two continuous function have the same integral w.r.t. all invariant regular probability measures iff their difference can be uniformly approximated by sums of functions of the form $f(\omega) - f(x\omega)$, $f(\omega)$ continuous, $x \in G$. Such functions are "close to being equidecomposable" via signed continuous functions. Our interest will lie, however, with equidecomposability via *nonnegative* functions, where the matters are a bit less simple. Yet, our main tools will still be theorems akin to convex separation, which are, of course, a part of the Hahn-Banach philosophy.

Remark 3.1.1 Equidecomposability of functions has been studied extensively by Friedrich Wehrung (see, for example, [We1], [We2], [We0]), using his algebraic methods concerning positively ordered monoids and in connection with the Banach-Tarski paradox (see [Wa]). He calls it *continuous* equidecomposability, "continuous" referring to the functions being $\overline{\mathbf{R}^+}$-valued, while the Banach-Tarski paradox about decomposability of sets refers to $\overline{\mathbf{Z}^+}$-valued functions. It seems that our methods and results, being more functional analytic, are somewhat different.

3.2 Upper Semi-Continuous and Baire Lower Semi-Continuous Functions

For the rest of Section 3 our groups will be discrete (not necessarily countable). Thus *amenable* will mean: *amenable as a discrete group.*

It seems better, when one wishes to speak about the totality of invariant measures, to deal with a compact or at least locally compact space. When a group G acts on it by homeomorphisms, we have a compact G-space. Note that G-measurable spaces can often be related to compact G-spaces (see §A.3, which deals with the continuous 2nd-countable case, containing the case of countable discrete G). By Stone's duality ([Ha-B]) G-Boolean algebras are equivalent to compact totally disconnected G-spaces.

Remark 3.2.1 Moreover, if a Boolean σ-field $\mathcal{B}$ of subsets of some set Ω is given, with a given fixed Boolean σ-ideal $\mathcal{J}$ of *null sets*, then many measure theoretic concepts in Ω correspond canonically to topological concepts in the Stone space $\mathcal{S}$ of $\mathcal{B}/\mathcal{J}$ (see [El]). In particular, a measure in $\mathcal{B}$ zero on $\mathcal{J}$ induces a measure in $\mathcal{S}$; bounded measurable functions from Ω to some metrizable compact space K, modulo null sets, are in 1-1 correspondence with continuous functions on $\mathcal{S} \to K$, (taking $K = [0, \infty]$ deals with unbounded functions), where convergence a.e. corresponds to convergence on $\mathcal{S}$ modulo meager sets.

Let us fix some notations.

By a **p.m. (probability measure)** we will mean a regular Borel probability measure on a compact or locally compact Ω.

We shall deal with **u.s.c. (upper semi-continuous)** functions and **l.s.c. (lower semi-continuous)** functions. Unless stated otherwise, u.s.c. functions will be assumed $[-\infty, \infty[$-valued and l.s.c. functions $]-\infty, \infty]$-valued.

Special attention will be given to nonnegative **Baire l.s.c. (b.l.s.c.)** functions in the sense of [Ha-M] Ch. X. These are nonnegative l.s.c. functions f s.t. the open $\{f > a\}$ is σ-compact for all $a \in \mathbf{R}^+$.

Remark 3.2.2 Note the following well-known facts, where all functions are real on a compact space Ω: ($f > g$ means $\forall \omega \in \Omega \;\; f(\omega) > g(\omega)$)

1. any u.s.c. function is the pointwise infimum of the continuous functions dominating it.
2. Integration w.r.t. a (positive) regular finite measure commutes with taking pointwise infima of a directed downward family of u.s.c. functions.
3. Any family, directed downwards w.r.t. $\leq$, of continuous functions whose infimum is f (necessarily u.s.c.) is cofinal downwards w.r.t. the continuous functions strictly $> f$.
4. If $(f_i)_i$ is a *finite* family of u.s.c. functions, then any continuous $h > \sum_i f_i$ can be written as $h = \sum_i h_i$, h_i continuous and $h_i > f_i$. (this follows from item 3, for the family of all h's expressible as such sums.)
5. Dual facts hold for l.s.c. functions.
6. The sum of a (not necessarily countable) family of *nonnegative* l.s.c. functions is l.s.c.
7. If a continuous function f is the sum of a (not necessarily countable) family of nonnegative l.s.c. functions (f_α), then each f_α is continuous (this follows from f_α being also u.s.c., since $f_\alpha = f - \sum_{\beta \neq \alpha} f_\beta$). Moreover, then the summation is uniform on compacta (Dini's Thm.)
8. Similar facts hold for functions on a locally compact Ω, where the u.s.c. (in particular, continuous) functions are required to have compact support, and for such function g one requires, instead of $f > g$, that $f > g$ *on* $\operatorname{supp} g$.
9. Any b.l.s.c. function is the supremum of a *countable* family of continuous functions with compact support (this family can be chosen monotone nondecreasing.)

Remark 3.2.3 Note that if $\mathbf{Z}$ is acting on a compact Ω and $E \subset \Omega$ is clopen, then ρ_E (see §1.1) is *b.l.s.c.*, this following from the fact that its 0-chain is a vertex of a 1-chain continuous in $\omega \in \Omega$. Note that by §A.3, for any standard Borel Ω on which $\mathbf{Z}$ acts and for any Borel $E \subset \Omega$, Ω can be embedded as a Borel subset in a totally disconnected compact $\bar{\Omega}$ with E an intersection of Ω with a clopen $\bar{E}$.

3.3 Two Theorems

One has the following two theorem, one dealing with u.s.c. functions and the other with b.l.s.c. functions. They will be presented here with some corollaries. The proofs are given in §3.4.

Theorem 3.3.1 *Let G be a discrete group (possibly uncountable). Let Ω be a G-compact space.*

Let $f : \Omega \to \mathbf{R}^+$ and $g : \Omega \to \mathbf{R}^+$ be nonnegative u.s.c. (as mentioned above they are assumed finite).

Suppose any G-invariant p.m. on Ω gives to f a greater or equal integral than to g.

Then every continuous function strictly greater than f is strictly greater than some u.s.c. function finitely decomposable with g via u.s.c. functions.

Equivalently (see Remark 3.2.2 item 4):

Every continuous function strictly greater than f is finitely equidecomposable via nonnegative continuous functions with some (continuous) function strictly greater than g.

The phrase: f' and f'' being "finitely equidecomposable via continuous functions" is self-explanatory: it means that $\exists$ a finite $I \subset G$ and nonnegative continuous functions f_x, $x \in I$ s.t. $f' = \sum_{x \in I} f_x$ while $f''(\omega) = \sum_{x \in I} f_x(x\omega)$ $\omega \in \Omega$. Alternatively, this may be expressed as: the 0-chains of f' and f'' are the two vertices of a continuous 1-chain (i.e. the coefficient of every 1-simplex depends continuously on ω) which is supported in a fixed finite union of sets of the form $\{(x_1 y, x_2 y) : y \in G\}$. Similar expressions will have similar meanings.

Remark 3.3.2 One cannot replace in Thm. 3.3.1 "continuous function strictly greater than f" by f itself. A simple example is given by $f \equiv 0$ and g being 0 except at a single point x_0 where it is 1, where x_0 has an infinite G-orbit.

Now, take as one of the functions in Thm. 3.3.1 a constant a, to obtain:

Corollary 3.3.3 *Let G be a discrete group (possibly uncountable). Let Ω be a G-compact space.*

Let f be nonnegative u.s.c. on Ω. If a constant a is greater than the supremum of the integrals of f w.r.t. all G-invariant p.m. then $\exists$ a function finitely equidecomposable with f via continuous (resp. u.s.c.) functions and dominated by a.

In other words, for a continuous (resp. u.s.c.) function f, the infimum of the maxima of all functions finitely equidecomposable with it via continuous (resp. u.s.c.) functions is equal to the supremum of the integrals of f w.r.t. G-invariant p.m.'s.

Clearly, in Cor. 3.3.3 one may replace "continuous" by "member of a fixed dense subalgebra $\mathcal{A}$ of $\mathcal{C}(\Omega)$ s.t. for any nonvanishing $f \in \mathcal{A}$ one has $1/f \in \mathcal{A}$". As $\mathcal{A}$ one may take the set of smooth functions (for Ω a compact manifold) of the set of continuous simple (i.e. with finite range) functions (for Ω compact totally disconnected). Since (see [Ha-B]) such Ω is equivalent, by Stone's duality, to a G-Boolean algebra $\mathcal{B}$, where every finitely additive p.m. on $\mathcal{B}$ can be uniquely extended to a σ-additive regular Borel p.m. om Ω, this leads to the following statement dealing with equidecomposition of "sets", in the spirit of the Banach-Tarski paradox (see [Wa]):

Corollary 3.3.4 *Let G be a discrete group (possibly uncountable). Let $\mathcal{B}$ be a G-Boolean algebra.*

Let $E \in \mathcal{B}$. If a constant a is greater than the supremum of the masses of E w.r.t. all G-invariant finitely additive p.m. on $\mathcal{B}$ then $\exists$ $p \in \mathbf{N}$ and a finite decomposition of p times E into members of $\mathcal{B}$, with G-translates of the pieces covering every part of 1 *no more than $\lfloor ap \rfloor$ times.*

In this, "Banach-Tarski paradox-like" context, one may inquire, for instance:

Can p be given "in advance"? That is, suppose q/p is greater than the above supremum. Are we sure p times E can be decomposed to pieces whose translates cover every part no more than q times? Note that $\mathcal{B}$ is here arbitrary.

What happens if q/p is *equal* to the above supremum? (or if in Cor. 3.3.3 "greater than the supremum" is replaced by "greater or equal"?)

Compare with Examples 3.5.8, 3.5.9 and 3.5.10.

Now turn to the theorem dealing with b.l.s.c. functions.

Theorem 3.3.5 *Let G be a discrete group (possibly uncountable). Let Ω be a G-locally compact space.*

Let $f : \Omega \to \overline{\mathbf{R}^+}$ and $g : \Omega \to \overline{\mathbf{R}^+}$ be nonnegative b.l.s.c. (as mentioned above, they may take the value $+\infty$.)

(i) suppose:

- *An orbit of G intersects the set $\{g > 0\}$ iff it intersects the set $\{f > 0\}$.*
- *If U is the (open) union of the orbits that intersect $\{g > 0\}$ and $\{f > 0\}$, then any (non-negative) G-invariant Radon measure[7] on U (possibly with infinite mass) gives the same integral to f and g.*

Then f and g are countably equidecomposable via nonnegative b.l.s.c. functions.

note: *if f is continuous, all the functions in the decomposition are continuous and the decomposition of f is uniformly convergent on compacta – see Remark 3.2.2 item 7.*

[7]Recall that a Radon measure on a locally compact space is a measure, with every open set measurable, finite on compact sets, and regular, in the sense that the mass of every measurable set is the infimum of the mass of opens containing it, and the mass of every open set is the supremum of the mass of compacts contained in it. If the locally compact space is 2nd-countable regularity for Borel sets is automatic, and the mass of every Borel set is the supremum of the mass of compacts contained in it as well as the infimum of the mass of opens containing it.

(ii) suppose:

- *Any orbit of G that intersects the set $\{g > 0\}$ intersects the set $\{f > 0\}$.*
- *If U is the (open) union of the orbits that intersect $\{g > 0\}$, then any (non-negative) G-invariant Radon measure on U (possibly with infinite mass) gives to f an integral greater or equal than to g.*

Then f is greater or equal than some function countably equidecomposable with g via nonnegative b.l.s.c. functions.

Example 3.3.6 From Thm. 3.3.5 one obtains that if Ω is a G-metrizable compact space, and $x_0 \in \Omega$ is a point with infinite orbit, then 1 and $1 - 1_{\{x_0\}}$ are countably equidecomposable (via l.s.c. functions) while $1 - 1 = 0$ and $1 - (1 - 1_{\{x_0\}}) = 1_{\{x_0\}}$ are not equidecomposable. Thus the relation of countable equidecomposability does not carry over to differences.

From Stone's duality point of view, the setting of Thm. 3.3.5 corresponds to $\overline{R^+} = [0, \infty]$-valued finitely additive measures μ on a G-Boolean algebra $\mathcal{B}$ with dual compact totally disconnected Stone space Ω. The union of clopens in Ω which, as members of $\mathcal{B}$, have finite μ-mass is an open, hence locally compact, $U \subset \Omega$ on which μ defines a unique Radon measure. Thus if $E \in \mathcal{B}$ then G-invariant $\overline{R^+}$-valued measures μ with $0 < \mu(E) < \infty$ correspond to G-invariant Radon measures μ on the open union of orbits in Ω that intersect the clopen E, s.t. $0 < \int 1_E \, d\mu < \infty$. The following corollary of Thm. 3.3.5 may be thought of as a "continuous" "topological" analog of Tarski's theorem which states, for $\mathcal{B}$ the field of subsets of some set, that such μ exists iff $2E$ is not equidecomposable (as sets) with E (see [Pa], [Ta])

Corollary 3.3.7 *Let G be a discrete group (possibly uncountable). Let Ω be a G-locally compact space.*

Let $f : \Omega \to \mathbf{R}$ be nonnegative b.l.s.c. (as mentioned above, it may take the value $+\infty$.)

Let U be the open union of the orbits which contain a point where f does not vanish.

If there is no G-invariant Radon measure on U that gives to f the integral 1, then for every $0 < a < \infty$ af and f are countably equidecomposable via b.l.s.c. functions.

(If f is continuous the functions in the decomposition are continuous and the decompositions converge uniformly on compacta.)

Proof Just apply Thm. 3.3.5 (i) to f and af.
QED

Thm. 3.3.5 can be applied to Stone-Čech compactifications of discrete sets:

Let a group G act (on the left) on a set S. Call a set $T \subset S$ **syndetic** if $\exists$ a finite $F \subset G$ s.t. $F \cdot T = \{xt : x \in F, \, t \in T\} = S$. Call a function $f : S \to \overline{R^+}$ **strictly syndetically supported** if for some $\varepsilon > 0$ the set $\{f \geq \varepsilon\}$ is syndetic.

Now consider the Stone-Čech compactification βS, which is a G-compact space. a function $f : S \to \overline{R^+}$ extends to a continuous function, to be denoted also by f, from βS to the compact $\overline{R^+}$, which is thus a b.l.s.c. function $f : \beta S \to \overline{R^+}$. If f is strictly syndetically supported, there is no G-invariant filter $\mathcal{F}$ on S s.t. $\lim_{\mathcal{F}} f = 0$, since such filter must contain the complements of all sets $x \cdot \{f \geq \varepsilon\}$, hence must contain $\emptyset$ since some $\{f \geq \varepsilon\}$ is syndetic. Thus there is no non-empty G-invariant closed set in βS on which f vanishes, hence the union of G-orbits intersecting $\{f > 0\}$ in βS is the whole βS. From Thm. 3.3.5 one can now deduce:

Corollary 3.3.8 *Let a group G act on a set S. Let $f, g : S \to \overline{R^+}$ be strictly syndetically supported.*

If f and g have the same integral w.r.t. all G-invariant finitely additive probability measures on S (where the integral of an unbounded function is defined as the supremum of the integrals of bounded functions majorized by it), then f and g are countably equidecomposable via non-negative functions on S, the decompositions converging uniformly on any set where the relevant sum is bounded.

If f has greater or equal integral than g w.r.t. any G-invariant finitely additive probability measure on S, then f is greater or equal than some function countably equidecomposable with g via non-negative functions on S, with the above uniform convergence property.

Proof Consider f and g extended to βS as continuous to $\overline{R^+}$. By the remarks preceding the statement of the corollary, the set U in Thm. 3.3.5 for f or g on βS is the whole βS. Hence Radon measures on $U = \beta S$ are finite, and they correspond to the finitely additive finite measures on S. Thus, by Thm. 3.3.5, f and g in our case are countably equidecomposable via b.l.s.c. functions on βS. The uniform convergence follows from Dini's Thm.

QED

Note that some requirement, such as f and g being strictly syndetically supported, is needed, as is shown by the example of $f \equiv 0$ and g s.t. for every $\varepsilon > 0$ $\{g > \varepsilon\}$ is a set with infinite number of disjoint translates (e.g. g tending to 0 at infinity).

For the case that G is amenable and the invariant probability measure is unique (e.g. a $\mathbf{Z}$ action by irrational rotation on the circle) one has

Corollary 3.3.9 *Suppose G is amenable.*

Let us be given a uniquely ergodic G-compact space Ω, i.e. a continuous action of G on Ω s.t. on Ω $\exists$ a unique invariant probability measure μ. (equivalently, $\exists$ a unique ergodic *invariant probability measure). Suppose, moreover, that the support of μ is Ω*

Then two b.l.s.c. non-negative functions that have the same μ-integral are countably equidecomposable via b.l.s.c. functions.

Proof Just apply Thm. 3.3.5, taking into account the fact that in our case every orbit is dense (otherwise its closure is an invariant proper subset, on which $\exists$ some invariant probability measure different from μ).

QED

Remark 3.3.10 When a standard G-space Ω is embedded as an (invariant) Borel subset in a metrizable compact G-space $\overline{\Omega}$, the totality of G-invariant measures in Ω is a subset of the totality of G-invariant measures in $\overline{\Omega}$, namely, those giving mass 0 to $\overline{\Omega}\backslash\Omega$. For example, when $\overline{\Omega}$ is constructed as a Stone space of an (invariant) countable Boolean Algebra $\mathcal{B}$ forming a basis to the Borel sets, any *finitely* additive (say, invariant) measure on $\mathcal{B}$ corresponds to a (σ-additive) measure in $\overline{\Omega}$, the latter being concentrated in Ω if and only if the original measure was σ-additive. One feels that considering invariant measures in the compact $\overline{\Omega}$ "includes measures that have flown away". On the other hand (see §A.3) for any countable family of Borel non-negative functions (resp. bounded Borel non-negative functions) on Ω there is a $\overline{\Omega}$ such that these functions can be extended to l.s.c. (resp. continuous) functions on $\overline{\Omega}$. One may say that when the theorems of this § are applied to an $\overline{\Omega}$ "also some flown-away measures are taken into account", while the resulting equidecomposability "holds also for limit values". They do not address the case when only (σ-finite) measures in Ω "proper" are considered.

3.4 Proofs

Proof of Thm. 3.3.1:

Introduce the following notation: If $\mathcal{A} \subset \mathcal{C}(\Omega)^+$, then $\tilde{\mathcal{A}}$ is the set of all $h \in \mathcal{C}(\Omega)$ finitely equidecomposable via nonnegative continuous functions with some $f \in \mathcal{A}$. For $A = \{f\}$ write $\tilde{f} := \tilde{A}$.

Lemma 3.4.1 *Let Ω be a compact G-space. Let μ be a positive measure on Ω. Then the functional $\tilde{\mu}$ on $\mathcal{C}(\Omega)$, corresponding to every $f \in \mathcal{C}(\Omega)^+$ the infimum of μ on $\tilde{f}$ is additive, so it extends to a bounded positive functional on $\mathcal{C}(\Omega)$, that is, to an (evidently invariant) finite positive regular measure $\tilde{\mu}$.*

Proof of Lemma 3.4.1: This follows from the fact that $\tilde{}(f+g) = \tilde{f} + \tilde{g}$. The latter is a consequence of $\mathcal{C}(\Omega)$ being an Abelian group lattice, hence has the *decomposition property* (or *refinement property* – see [We1]): If $f_i, g_j \geq 0$ $i = 1, \ldots, n$ $j = 1, \ldots, m$ and $\sum_i f_i = \sum_j g_j$ then $\exists h_{ij} \geq 0$ s.t. $f_i = \sum_j h_{ij}$, $g_j = \sum_i h_{ij}$ – see [Bo-A] Ch. VI §1 Thm. 1.

QED

Continuation of the Proof of Thm. 3.3.1: For our u.s.c. function f, denote by $\mathcal{H}_f$ the set of continuous functions $> f$. Define $\mathcal{H}_g$ analogously.

A rephrasing of the theorem is (see Rmk. 3.2.2): if $\langle \nu, f\rangle \geq \langle \nu, g\rangle$ for any G-invariant p.m. ν on Ω, then $\mathcal{H}_f \subset \tilde{\mathcal{H}_g}$. Recall that $\tilde{\mathcal{H}_g}$ is the set of continuous functions $>$ some function finitely equidecomposable with g via u.s.c. functions.

It is easily proved that $\tilde{\mathcal{H}_g}$ is convex. It is also open in $\mathcal{C}(\Omega)$. Indeed, if $h \in \mathcal{H}_g$, then since $g-h$ is u.s.c. <0, $\max(g-h)<0$, thus for some $\varepsilon>0 \quad h-\varepsilon$ is still $>g$, hence for any $h' \in \tilde{h}, \quad h'-\varepsilon \in \tilde{\mathcal{H}_g}$.

Thus one can apply separation theorems for open convex sets. Suppose $f_0 \in \mathcal{H}_f$ but $f_0 \notin \tilde{\mathcal{H}_g}$. Then $\exists$ a bounded functional $\mu \neq 0$ on $\mathcal{C}(\Omega)$ with $\mu > \mu(f_0)$ on $\tilde{\mathcal{H}_g}$. Since $\tilde{\mathcal{H}_g} + \mathcal{C}(\Omega)^+ \subset \tilde{\mathcal{H}_g}$, $\mu \geq 0$ and one may assume μ a p.m. Consider $\tilde{\mu}$ (Lemma 3.4.1). It is an invariant positive measure dominated by μ. By definition $\tilde{\mu} \geq \mu(f_0)$ on $\mathcal{H}_g$, hence $\tilde{\mu}(g) \geq \mu(f_0)$ (Remark 3.2.2 items 1 and 2). But we have $f_0 \in \mathcal{H}_f$, implying, as we saw above, $f_0 - \varepsilon > f$ for some $\varepsilon > 0$, hence $\tilde{\mu}(g) \geq \mu(f_0) > \mu(f) \geq \tilde{\mu}(f)$. This contradicts the assumption of the theorem, that $\langle \nu, f\rangle \geq \langle \nu, g\rangle$ for any G-invariant p.m. ν on Ω, *a fortiori* for any invariant finite positive ν.
QED

Proof of Thm. 3.3.5 We start with an analog of Thm. 3.3.1. Denote by $\mathcal{C}_{00}(\Omega)$ the vector lattice of real continuous functions on Ω with compact support. Endow $\mathcal{C}_{00}(\Omega)$ with the (Hausdorff) locally convex topology which is the direct limit of the spaces, for compact $K \subset \Omega$, $\mathcal{C}(K) \cap \mathcal{C}_{00}$ (normed by supremum norm on K). This is the strongest locally convex topology making all the inclusion maps from these spaces continuous (see [Bo-E]). The positive cone of the dual space $\mathcal{C}_{00}(\Omega)^*$ is identified with the set of Radon measures on Ω (see [Bo-I]).

Lemma 3.4.2 *Let G be a discrete group (possibly uncountable). Let Ω be a G-locally compact space. Let $f : \Omega \to \mathbf{R}$ and $g : \Omega \to \mathbf{R}$ be nonnegative l.s.c. (as mentioned above they may take the value $+\infty$). Suppose*

- *Any orbit of G intersects the set $\{g > 0\}$ and the set $\{f > 0\}$.*
- *Any (non-negative) G-invariant Radon measure on Ω (possibly with infinite mass) gives to f an integral greater or equal than to g.*

Then every continuous function with compact support which is strictly dominated on its support by g is finitely equidecomposable via nonnegative continuous functions with compact support with some function strictly dominated on its support by f.

Let us first show how the theorem follows from Lemma 3.4.2:

(i) We may and do assume $U = \Omega$ (note U is open in a locally compact space hence is itself locally compact).

Denote by $\sim$ the relation: "finitely equidecomposable via nonnegative continuous functions with compact support".

Write $f = \sup f_n$ $g = \sup g_n$ where $f_n, n = 1, 2, \ldots$ and $g_n, n = 1, 2, \ldots$ are nondecreasing sequences of continuous functions with compact support, with $f_n < f$ on $\operatorname{supp} f_n$, $g_n < g$ on $\operatorname{supp} g_n$ (see Remark 3.2.2 item 9).

By Lemma 3.4.2, $f_1 \sim g_1'$, g_1' being strictly dominated on its support by g. Replace $g_n, n \geq 1$ by $\max(g_n, g_1')$. Again by Lemma 3.4.2, $g_1 - g_1' \sim f_1' - f_1$, f_1' being strictly dominated on its support by f. Replace $f_n, n \geq 2$ by $\max(f_n, f_1')$. Again, $f_2 - f_1' \sim g_2' - g_1$, g_2' being strictly dominated on its support by g. Replace $g_n, n \geq 2$ by $\max(g_n, g_2')$. Now $g_2 - g_2' \sim f_2' - f_2$, and we continue, addressing alternatively f and g, *ad infinitum*. We have

$$f = f_1 + (f_1' - f_1) + (f_2 - f_1') + (f_2' - f_2) + \ldots$$
$$g = g_1' + (g_1 - g_1') + (g_2' - g_1) + (g_2 - g_2') + \ldots$$

and the terms are mutually finitely equidecomposable via nonnegative continuous functions with compact support. This means that they can be written as sums $\sum k_x$ and $\sum x k_x$, k_x continuous with compact support. Collecting all k_x with the same x for all the terms, we have our conclusion that f and g are countably equidecomposable via nonnegative b.l.s.c. functions.

(ii) The proof here follows the same lines as for item (i), but instead of addressing f and g alternatively we go in one direction, writing g as a series of functions $\sim$ to a series dominated by f.

The **Proof of Lemma 3.4.2** has similarities with the proof of Thm. 3.3.1. As we did there, introduce here the notation: If $\mathcal{A} \subset \mathcal{C}_{00}(\Omega)^+$, then $\tilde{\mathcal{A}}$ is the set of all $h \in \mathcal{C}_{00}(\Omega)^+$ finitely equidecomposable via nonnegative continuous functions with compact support with some $f \in \mathcal{A}$. For $A = \{f\}$ write $\tilde{f} := \tilde{A}$.

Let μ be a positive Radon measure on Ω. In analogy with Lemma 3.4.1, consider the functional $\tilde{\mu}$ on $\mathcal{C}_{00}(\Omega)^+$, corresponding to every $f \in \mathcal{C}_{00}^+$ the *supremum* of μ on $\tilde{f}$, In analogy with the proof of that lemma, μ (which may take the value $+\infty$) is additive. So, if it is finite and bounded on bounded sets in $\mathcal{C}_{00}^+$ (these are sets of functions bounded uniformly on every compact), it extends to a positive continuous functional on $\mathcal{C}_{00}(\Omega)$, i.e. an (invariant) positive Radon measure $\tilde{\mu}$.

For our l.s.c. function f, denote by $\mathcal{M}_f$ the set of non-negative continuous functions with compact support, strictly dominated on their support by f. Define $\mathcal{M}_g$ analogously.

A rephrasing of Lemma 3.4.2 is: if $\langle \nu, f\rangle \geq \langle \nu, g\rangle$ for any G-invariant Radon measure ν on Ω, then $\mathcal{M}_g \subset \tilde{\mathcal{M}_f} - \mathcal{C}_{00}^+$.

It is easily proved that $\tilde{\mathcal{M}_f} - \mathcal{C}_{00}^+$ is convex. Let us prove that it is open in $\mathcal{C}_{00}(\Omega)$ (note that $\mathcal{M}_f - \mathcal{C}_{00}^+$ need not be open – 0 is not an interior point if f is not always > 0). Indeed if $h \in \mathcal{M}_f$, then $f - h$ is l.s.c., > 0 on $\operatorname{supp} h$ hence on $\{f > 0\}$. We have $h + \tilde{\mathcal{M}_{f-h}} \subset \tilde{\mathcal{M}_f}$. Thus to prove $\tilde{\mathcal{M}_f}$ open it suffices to prove

Lemma 3.4.3 *For f nonnegative l.s.c. on Ω, s.t. any orbit of G intersects the set $\{f > 0\}$, the function 0 is an interior point of $\tilde{\mathcal{M}_f} - \mathcal{C}_{00}^+$.*

Proof of Lemma 3.4.3 By the definition of the topology in $\mathcal{C}_{00}(\Omega)$, we need to prove that $\forall$ compact $K \subset \Omega$ $\exists \varepsilon > 0$ s.t. all continuous functions h with support $\subset K$ and $\max |h| < \varepsilon$ belong to $\tilde{\mathcal{M}_f} - \mathcal{C}_{00}^+$.

The assumption that any orbit intersects $\{f > 0\}$ implies that the open sets $x \cdot \{u > 0\}$, $x \in G, u \in \mathcal{M}_f$ cover K. Choose a finite subcovering $x_j \cdot \{u_j > 0\}$, $j = 1, \dots N$ and let $u(\omega) = \frac{1}{N} \sum_j u_j(x_j^{-1}\omega)$. Then $u \in \tilde{\mathcal{M}_f}$ and $u > 0$ on K. Now take $\varepsilon = \min_{\omega \in K} u(\omega)$.

QED

This allows us to use separation theorems for open convex sets in the locally convex space $\mathcal{C}_{00}(\Omega)$. Suppose $f_0 \in \mathcal{M}_g$ but $f_0 \notin \tilde{\mathcal{M}_f} - \mathcal{C}_{00}^+$. Then $\exists$ a continuous functional $\mu \neq 0$ on $\mathcal{C}(\Omega)$ with $\mu < \mu(f_0)$ on $\tilde{\mathcal{M}_f} - \mathcal{C}_{00}^+$. This implies $\mu \geq 0$. We wish to consider $\tilde{\mu}$ (as defined above). We can assert that it is finite and continuous, since by Lemma 3.4.3, the fact that $\mu < \mu(f_0)$ on $\tilde{\mathcal{M}_f}$ implies $\tilde{\mu}$ is bounded above on $\tilde{\mathcal{M}_f} - \mathcal{C}_{00}^+$, a neighbourhood of 0. Thus $\tilde{\mu}$ is an invariant (positive) Radon measure on Ω and it is not 0 since $\mu \leq \tilde{\mu}$. By definition $\tilde{\mu} \leq \mu(f_0)$ on $\mathcal{M}_f$, hence $\tilde{\mu}(f) \leq \mu(f_0)$ (f is the pointwise supremum of $\mathcal{M}_f$ and a Radon measure commutes with suprema of l.s.c. functions – see Remark 3.2.2). We have $\tilde{\mu}(g - f_0) > 0$. Indeed, $f_0 \in \mathcal{M}_g$, hence $g - f_0 > 0$ on $\{g > 0\}$. Thus by Lemma 3.4.3 $\tilde{\mathcal{M}_{g-f_0}} - \mathcal{C}_{00}^+$ is a neighbourhood of 0, in which $\tilde{\mu}$ is dominated by $\tilde{\mu}(g - f_0)$, hence the latter is > 0. Thus $\tilde{\mu}(f) \leq \mu(f_0) \leq \tilde{\mu}(f_0) < \tilde{\mu}(g)$. This contradicts the assumption of the lemma, that $\langle \nu, f\rangle \geq \langle \nu, g\rangle$ for any G-invariant Radon measure ν on Ω.

QED

3.5 Averages

The simplest case of equidecomposability is that of a function and its averages.

Definition 3.5.1 *Given a G-vector space V (always viewed as real), (that is, the group G acts on V linearly). Let $v \in V$. An* **average** *of v is any element of V of the form*

$$\sum_{x \in G} \lambda_x x v \quad \lambda_x \geq 0, \sum_{x \in G} \lambda_x = 1, \text{ all but finitely many } \lambda_x = 0.$$

That is, a member of the convex hull of the orbit of v.

One may ask: to what extent can we characterize, say, equality of integral w.r.t. all invariant measures, allowing just averages instead of any equidecomposable function? In what sense are equidecomposable functions "similar" already in averages?

We use the following theorem, which is an infinite-dimensional version of the famous Von Neumann Minimax Theorem in Game Theory ([vN], [vNM], a standard textbook is [Ow]). For completeness, we give a proof:

Theorem 3.5.2 *Let Ω be a compact space.*

Let there be given a convex set $\mathcal{F}$ of u.s.c. functions on Ω.

Denote by $\mathcal{M}\Omega$ the set of all p.m.'s on Ω.

Then

$$\inf_{f\in\mathcal{F}} \max_{\omega\in\Omega} f(\omega) = \max_{\mu\in\mathcal{M}\Omega} \inf_{f\in\mathcal{F}} \langle \mu, f\rangle .$$

Proof Clearly

$$\inf_{f\in\mathcal{F}} \max_{\omega\in\Omega} f(\omega) \geq \max_{\mu\in\mathcal{M}\Omega} \inf_{f\in\mathcal{F}} \langle \mu, f\rangle .$$

Now let $a < \inf_{f\in\mathcal{F}} \max_{\omega\in\Omega} f(\omega)$ and we prove $a \leq \max_{\mu\in\mathcal{M}\Omega} \inf_{f\in\mathcal{F}} \langle \mu, f\rangle$.

W.l.o.g. we may assume $a = 0$ (replace each f by $f - a$). Let

$$\varepsilon = \inf_{f\in\mathcal{F}} \max_{\omega\in\Omega} f(\omega) > a = 0.$$

Let $\mathcal{H}$ be the set of all continuous functions dominating some member of $\mathcal{F}$. Then the function 0 is distanced at least ε from $\mathcal{H}$ in the sup-norm. Moreover, $\mathcal{H}$ is convex since $\mathcal{F}$ is so. By convex separation in the Banach space $\mathcal{C}(\Omega)$, $\exists$ a $\mu_0 \in \mathcal{C}(\Omega)^*$, $\mu \neq 0$ positive on $\mathcal{H}$. Since for nonnegative continuous $f \quad \mathcal{H} + f \subset \mathcal{H}$, μ_0 must be nonnegative on nonnegative continuous functions. Hence we may and do assume $\mu_0 \in \mathcal{M}\Omega$. By construction μ_0 is nonnegative on any continuous function dominating some $f \in \mathcal{F}$. By Remark 3.2.2 we have μ_0 nonnegative on any $f \in \mathcal{F}$, i.e. $\inf_{f\in\mathcal{F}} \langle \mu_0, f\rangle \geq 0$, implying $0 \leq \max_{\mu\in\mathcal{M}\Omega} \inf_{f\in\mathcal{F}} \langle \mu, f\rangle$.

QED

In case Ω is a compact convex space, every p.m. μ on Ω has a barycenter ω, and the integral over μ of a continuous (or u.s.c.) *affine* function f on Ω is $f(\omega)$ (see [Ph]). Thus 3.5.2 takes the form:

Corollary 3.5.3 *Let Ω be a compact convex space.*

Let there be given a convex set $\mathcal{F}$ of affine *u.s.c. functions on Ω.*

Then

$$\inf_{f\in\mathcal{F}} \max_{\omega\in\Omega} f(\omega) = \max_{\omega\in\Omega} \inf_{f\in\mathcal{F}} f(\omega).$$

Now let us return to averages in a G-vector space. Suppose a sublinear functional $p : V \to \mathbf{R}$ *invariant under* G is given. By sublinear is meant, as usual, that p satisfies:

$$p(u + v) \leq p(u) + p(v) \quad u, v \in V$$
$$p(\lambda v) = \lambda p(v) \quad v \in V,\ \lambda > 0$$

An example is: a G-compact space Ω, V an invariant subspace of $\mathcal{C}(\Omega)$, $p(f) = \max_{\omega\in\Omega} f(\omega)$.

In fact, the general case of invariant sublinear functionals reduces to the above example.

Indeed, Suppose p is invariant sublinear as above. Let Ω be the set of all linear functionals $\omega : V \to \mathbf{R}$ satisfying $\omega \leq p$, with the weak V-topology. Since

$$\omega \leq p \Rightarrow \omega(v) = -\omega(-v) \geq -p(-v)$$

Ω is a G-compact space, and obviously any $v \in V$ induces a continuous affine function v on Ω. By Hahn-Banach $p(v) = \max_{\omega\in\Omega} v(\omega)$.

Theorem 3.5.4 *Given a G-vector space V. Let $p : V \to \mathbf{R}$ be sublinear and G-invariant.*
Let $v \in V$, and denote by $\mathcal{AV}(v)$ the set of its averages (i.e. the convex hull of its orbit).
Denote by Ω the set of all linear functionals $\omega : V \to \mathbf{R}$ satisfying $\omega \leq p$.
Then

a. *Suppose G* amenable*. Then the infimum of p over $\mathcal{AV}(v)$ is equal to the maximum of $\langle \omega, v \rangle$ for all the G-invariant $\omega \in \Omega$.*

b. *For general G, the infimum of p over $\mathcal{AV}(v)$ is equal to the maximum over $\omega \in \Omega$ of $\inf_{x \in G} \langle x\omega, v \rangle = \inf_{x \in G} \langle \omega, xv \rangle$, i.e. to the maximum of the infimum of v on nonempty (convex) G-invariant subsets of Ω.*

Proof Note first, that the fact that we indeed get maxima (and not just suprema) follows from upper-semi-continuity of the maximized function on the compact Ω.

In case G is amenable, any compact convex invariant subset of Ω contains an invariant point. Hence a. follows from b.

To prove b., identify V with a set of affine continuous functions on the compact Ω. Then $p(v) = \max v$, and use Cor. 3.5.3:

The infimum of ($p = \max$) over the convex $\mathcal{AV}(v) =$
The maximum over $\omega \in \Omega$ of $\inf_{w \in \mathcal{AV}(v)} w(\omega) =$
The maximum over $\omega \in \Omega$ of $\inf_{x \in G} (xv)(\omega)$

and we are done.
QED

Corollary 3.5.5 *In case G is* amenable *(retaining the assumptions of the theorem):*
Two vectors giving the same value to all invariant $\omega \in \Omega$, in particular two vectors finitely equidecomposable (via elements of V), have the same infimum of p over the set of their averages.

Cor. 3.5.5 says that, in the amenable case, the property of two vectors to be finitely equidecomposable can be judged merely by their averages. (In case V is a G-normed space, this extends to one vector approximable by vectors finitely equidecomposable with another).

Cf. Example A.4.4 in §A.4.

Remark 3.5.6 Compare Thm. 3.5.4 a., for the case of $\mathcal{C}(\Omega)$ and $p = \max$, with Cor. 3.3.3. In the amenable case, functions finitely equidecomposable with a given function are replaced by its averages, on which there is much greater control.

Recall that Cor. 3.3.3 led to Cor. 3.3.4 for G-Boolean algebras. For G amenable, Thm. 3.5.4 gives:

Corollary 3.5.7 *Let G be* amenable *and let $\mathcal{B}$ be a G-Boolean algebra.*
Let $E \in \mathcal{B}$. If a constant a is greater than the supremum of the masses of E w.r.t. all G-invariant finitely additive p.m. on $\mathcal{B}$ then $\exists$ a $p \in \mathbf{N}$ and p G-translates of E, covering every part of 1 no more than $\lfloor ap \rfloor$ times.

Thus E is in **"generalized Rokhlin position"**, where an $E \in \mathcal{B}$ is in **"Rokhlin position"** if $\exists$ a finite set $\{x_1, x_2, \ldots, x_N\} \subset G$ s.t. x_iE, $i = 1, \ldots, N$ are disjoint. Then we are sure that for every invariant finitely additive p.m. on $\mathcal{B}$, the mass of $E \leq 1/N$. Similarly, If every part is covered by no more than k of the x_iE, $i = 1, \ldots, N$ then we know that for every such invariant p.m. $\mu \quad \mu(E) \leq k/N$.

Cor. 3.5.7 is a kind of converse statement.

Example 3.5.8 In Cor. 3.5.7 (hence in Cor. 3.5.5) the assumption that G is amenable is essential. Just let G be non-amenable (discrete) and take $\mathcal{B} = \mathcal{P}(G)$, G acting on itself by left translations, $E = 1 (= G) \in \mathcal{B}$. Note that the supremum of masses of invariant finitely-additive p.m.'s is 0, since no such p.m. exists.

Example 3.5.9 Cor. 3.5.7 with a *equal* to the supremum of the masses (and Thm. 3.5.4 a. with "infimum" over $\mathcal{AV}(v)$ replaced by "minimum") may not hold. As a counterexample take $G = \mathbf{Z}$ acting on itself by translations, $\mathcal{B} = \mathcal{P}(\mathbf{Z})$, $E = 2\mathbf{Z} \cup \{1\}$.

Example 3.5.10 In Cor. 3.5.7 p may not be chosen "in advance": Let $G = C_n^2$, C_n being cyclic of order n. Let G act on itself by translations and let $\mathcal{B} = \mathcal{P}(G)$, $E = (C_n \times \{e\}) \cup (\{e\} \times C_n)$. Although the unique G-invariant p.m. gives to E a measure $< 2/n$, E is not in "Rokhlin position" even for $N = 2$, since it intersects each of its translates.

See §A.4 for a treatment of "mean ergodic theorems" for general (discrete) groups, where "convergence" means: "having averages arbitrarily near the limit".

4 Equidecomposable Enhanced Functions

4.1 Introduction and Inquiry

In a similar manner to what was said in Section 3, in the continuous case too some notions are independent of the invariant measure μ in an Ω acted by G. Such are m-chains and their vertices, in particular enhanced functions. Two enhanced functions will be called **equidecomposable**, with suitable qualifications (such as, in the discrete case: finitely-, countably-, via continuous functions etc.) if a suitable 1-chain exists, s.t. the two enhanced functions are, resp., its source and target. This notion too does not depend on the invariant measure μ in Ω.

Note that 1-chains obtained from weighted hypergraphs are not sufficient: even to have one enhanced function equidecomposable with itself one needs a 1-chain supported on the diagonal of G^2.

Thus, in applications of CHG, such as those in Section 2, in particular §2.8, the two enhanced functions are, in fact, *equidecomposable* (via measurable 1-chains).

CVE tells us that equidecomposable enhanced functions have the same expectation, in other words if two functions, enhanced by Λ_1 and Λ_2 resp., are equidecomposable, then *for any invariant probability measure μ in Ω*, their integrals w.r.t. the corresponding infinitesimal measures are the same.

Thus the relation of two functions to be equidecomposable after enhancement by Λ_1 and Λ_2 appears as *dual* to the relation of two measures to be infinitesimal measures obtained from the same invariant probability μ by Λ_1 and Λ_2. One may try to take this duality as a *defining property* of one of these two notions starting from the other.

To this end, one may try to prove analogs of the facts in Section 3 for *enhanced functions*. It may be better to work in topological G-spaces, and to restrict oneself to enhanced functions which give u.s.c. or l.s.c. 0-chains, and also to equidecomposability understood as being the source and target of a u.s.c. or l.s.c. 1-chain. One may define such attributes of chains by requiring that applying them to every nonnegative continuous with compact support *test-function* be u.s.c. or l.s.c. In this setting, does one has an analog to Thm. 3.3.5?

Given a Borel-measurable m-chain (this means, as usual, that applying it to a test-function on G^{m+1} depends on ω in a Borel manner). Choose a countable collection $\mathcal{F}$ of test-functions (i.e. non-negative continuous with compact support) on G^{m+1} s.t. *every test-function is a non-decreasing limit of a sequence of members of $\mathcal{F}$* and s.t. *every member of $\mathcal{F}$ is a positive combination of convolutions of two test-functions* – see Rmk. 2.2.1. By §A.3 we may assume Ω is a dense Borel subset of a G-metrizable compact space K s.t. all applications of the m-chain to members of $\mathcal{F}$, hence to all test-functions, extend to l.s.c. functions on K. (The application to a convolution of two test-function is a convolution of a non-negative Borel function on Ω and a test-function (i.e. L^1-function) on G, and by §A.3 any countable collection of such extend to l.s.c. for a suitable K.) Thus the m-chain is l.s.c. on Ω with the relative Lusin topology (see §A.2). Can the chain be extended to the compact G-space K ? Then it would be extendable to a b.l.s.c. m-chain on the canonical non-metrizable compactification $\mathcal{K}$ in §A.3. Can this extension be made unique? Can this be used for the purposes mentioned above?

Before dealing with such questions, there is the question of the *transitivity* of the (qualified) equidecomposability relation, even for the discrete case. This can be put as follows: suppose two 1-chains have a common vertex (that is, a common projection on G); are the other two vertices the two vertices of some 1-chain? or, if we are lucky, does there exist a 2-chain having the given 1-chains as "sides", i.e. projections on G^2? In the discrete case such a 2-chain can be constructed, using some canonical construction of a nonnegative matrix with given row- and column- sums. But qualifications such as u.s.c. may be violated. Another approach to the transitivity is via criteria for equidecomposability such as Thm. 3.3.5. §4.2 below refers to the question of transitivity.

Peculiar to the continuous case are question such as: given Λ_1 and Λ_2 and f_1. Does there exist an f_2 s.t. f_1 enhanced by Λ_1 is equidecomposable with f_2 enhanced by Λ_2?, and a related question: can one reconstruct μ from the infinitesimal measure $\frac{d\Lambda}{d\lambda}\mu$? This is dealt with in §4.3 below.

4.2 Tame Invariant Chains and Transitivity of Equidecomposability

Our setting is a 2nd-countable locally compact group G acting in a Borel manner on a standard Borel space Ω. In this §, *measurable* will mean either "Borel measurable" or "universally measurable". We use the notions about chains (in the "continuous" case) mentioned in §2.2. For simplicity, we restrict ourselves to the case that G is *unimodular*, thus speak of invariant chains. For non-unimodular G, replace "invariant" by "Δ-right-invariant" and "Haar measure" by "left Haar measure".

Definition 4.2.1 *A (non-negative) invariant (ω-dependent) m-chain is called (Borel measurably, resp. universally measurably)* **tame** *if it can be written as a countable sum of non-negative measurable invariant chains all of which are Radon measures (on G^{m+1}) for each $\omega \in \Omega$ (i.e. are finite on compacta) – the latter will be referred to as (measurable, invariant)* **Radon chains***.*

If two 0*-chains are source and target of the same* tame 1*-chain, we say that they are* **Borel-tamely***- resp.* **universally measurably tamely equidecomposable***.*

Check that in most of the examples in Section 2 the chains are tame.

A vertex, and more generally a face, of a tame m-chain is tame. This follows from the fact that an invariant measurable Radon chain ϕ is a countable sum of invariant measurable Radon chain with Radon projections – just write $d\phi = \sum_n h_n(x_0x_1^{-1}, \ldots, x_0x_m^{-1})\, d\phi$ where the h_n are test-functions on G^m with sum $\equiv 1$.

Remark 4.2.2 We shall use the fact that if ν_n is a sequence of Radon measures in a 2nd-countable locally compact X, then there exists a sequence $\alpha_n > 0$ s.t. $\sum_n \alpha_n\nu_n$ is a Radon measure – just take an increasing sequence of compact sets $K_n \subset X$ that eventually contains any fixed compact $\subset X$, and choose $\alpha_n > 0$ s.t. $\sum_n \alpha_n\nu_n(K_n) < \infty$. Note that all the ν_n are absolutely continuous w.r.t. $\sum_n \alpha_n\nu_n$, with bounded Radon-Nikodym derivatives.

Lemma 4.2.3 *Let ϕ_1 and ϕ_2 be measurable invariant (ω-dependent) Radon chains. Then the (invariant) chain $\phi_1 \wedge \phi_2$ mapping each $\omega \in \Omega$ to the infimum of $\phi_1(\omega)$ and $\phi_2(\omega)$ in the lattice of non-negative Radon measures is also measurable.*

Proof The lemma follows from $\langle \phi_1(\omega) \wedge \phi_2(\omega), h\rangle$ (h – a *test-function* on G^{m+1}, i.e. a non-negative continuous function with compact support) being equal to the infimum of $\langle \phi_1, h_1\rangle + \langle \phi_2, h_2\rangle$ over the pairs $\{(h_1, h_2) : h_1, h_2 \in \mathcal{F},\ h_1 + h_2 \geq h\}$, $\mathcal{F}$ being a countable collection of test-functions, s.t. every test-function is the limit of a non-decreasing sequence of members of $\mathcal{F}$ (see Rmk. 2.2.1).

QED

Lemma 4.2.4 *Let ϕ_0, ϕ_1 and ϕ_2 be measurable invariant (ω-dependent) Radon chains, s.t. for each $\omega \in \Omega$ ϕ_1 and ϕ_2 are absolutely continuous w.r.t. ϕ_0. Then the (invariant) chain ϕ_3, s.t. for each ω*

$$d\phi_3 := \frac{d\phi_1}{d\phi_0}\frac{d\phi_2}{d\phi_0}\, d\phi_0 \tag{35}$$

is also measurable.

Proof If we had in (35) $\wedge$, i.e. min instead of multiplication of the Radon-Nikodym derivatives, the assertion would follow from Lemma 4.2.3. Therefore we would be done if we show how to express multiplication in $\overline{\mathbf{R}^+}$ using $\wedge$ and "well-behaved" limits (i.e. which preserve the measurability of the chain). This is done by ($a, b \in \overline{\mathbf{R}^+}$):

$$ab = \lim_n 2^{-n} \sum_{j,k \geq 1} \left[(a \wedge (j \cdot 2^{-n})) - (a \wedge ((j-1) \cdot 2^{-n}))\right] \wedge \left[(b \wedge (k \cdot 2^{-n})) - (b \wedge ((k-1) \cdot 2^{-n}))\right]$$

the limit being uniform in a, b.

(To convince oneself of the validity of this formula, note that for all except one of the summands, one of the expressions in [ ] is 2^{-n}, the other being some $t \in [0, 2^{-n}]$, and then $2^{-n}(2^{-n} \wedge t) = 2^{-n}t$.)

QED

Lemma 4.2.5 *Let $(\phi_i)_{i\geq 1}$ and $(\phi'_j)_{j\geq 1}$ be two sequences of measurable invariant Radon m-chains s.t. $\sum_i \phi_i = \sum_j \phi'_j$. Then one can find measurable invariant Radon m-chains ϕ_{ij} s.t. $\phi_i = \sum_j \phi_{ij}$ and $\phi'_j = \sum_i \phi_{ij}$.*

Proof For $i \geq 0$, $j \geq 0$ write $\psi_i := \sum_{1\leq \ell \leq i} \phi_\ell$, $\psi'_j := \sum_{1\leq \ell\leq j} \phi'_\ell$. Let $\Psi := \sum_i \phi_i = \sum_j \phi'_j$. Then $\psi_i \uparrow \Psi$, $\psi'_j \uparrow \Psi$, where we have this limit relation for the application of the measures to any $[0,\infty]$-valued Borel function on G^{m+1}. Construct a matrix of chains $(\phi_{ij})_{i,j\geq 1}$ s.t. (cf. Lemma 4.2.3)

$$\sum_{1\leq \ell\leq i, 1\leq \ell'\leq j} \phi_{\ell\ell'} = \psi_i \wedge \psi_j \tag{36}$$

in other words,

$$\phi_{ij} = \psi_i \wedge \psi'_j + \psi_{i-1} \wedge \psi'_{j-1} - \psi_i \wedge \psi'_{j-1} - \psi_{i-1} \wedge \psi'_j$$

The measurability of ψ_{ij} follows from Lemma 4.2.3 provided we prove that the ψ_{ij} are *non-negative*, i.e. that

$$\psi_i \wedge \psi'_{j-1} + \psi_{i-1} \wedge \psi'_j \leq \psi_i \wedge \psi'_j + \psi_{i-1} \wedge \psi'_{j-1}.$$

That follows from the following chain of assertions:

$$\begin{aligned}
&\psi_i \wedge \psi'_{j-1} + \psi_{i-1} \wedge \psi'_j \leq \psi_i + \psi_{i-1}\\
&\psi_i \wedge \psi'_{j-1} + \psi_{i-1} \wedge \psi'_j \leq \psi_i + \psi'_{j-1}\\
&\psi_i \wedge \psi'_{j-1} + \psi_{i-1} \wedge \psi'_j \leq \psi'_j + \psi_{i-1}\\
&\psi_i \wedge \psi'_{j-1} + \psi_{i-1} \wedge \psi'_j \leq \psi'_j + \psi'_{j-1}\\
&\psi_i \wedge \psi'_{j-1} + \psi_{i-1} \wedge \psi'_j \leq \psi_i + \psi_{i-1} \wedge \psi'_{j-1}\\
&\psi_i \wedge \psi'_{j-1} + \psi_{i-1} \wedge \psi'_j \leq \psi'_j + \psi_{i-1} \wedge \psi'_{j-1}\\
&\psi_i \wedge \psi'_{j-1} + \psi_{i-1} \wedge \psi'_j \leq \psi_i \wedge \psi'_j + \psi_{i-1} \wedge \psi'_{j-1}
\end{aligned}$$

Being dominated by Radon chains, ϕ_{ij} are Radon chains.

It remains to prove the claims $\sum_j \phi_{ij} = \phi_i$ and $\sum_i \phi_{ij} = \phi'_j$. By (36) these are equivalent to $\lim_j \psi_i \wedge \psi'_j = \psi_i$, $\lim_i \psi_i \wedge \psi'_j = \psi'_j$. Since $\lim_i \psi_i$ and $\lim_i \psi'_j$ are Ψ, and $\psi_i \wedge \Psi = \psi_i$, $\Psi \wedge \psi'_j = \psi'_j$, all we have to show is that in our case lim commutes with $\wedge$. This can be shown for each ω separately. By Rmk. 4.2.2, for any fixed ω there is a Radon measure s.t. all the ϕ_i and ϕ'_j are absolutely continuous w.r.t. it, with bounded Radon-Nikodym derivatives. Thus to prove lim commutes with $\wedge$ we can pass to the Radon-Nikodym derivatives, which are functions, and for them this is immediate.

QED

Lemma 4.2.6 *Let $(\phi_i)_{i\geq 1}$ be a sequences of measurable invariant Radon m-chains s.t. a certain k-dimensional face $(0 \leq k \leq m)$ of the sum $\sum_i \phi_i$ can be written as $\sum_j \phi'_j$ for some measurable invariant Radon k-chains $(\phi'_j)_{j\geq 1}$. Then one can find measurable invariant Radon k-chains ϕ_{ij} s.t. $\phi_i = \sum_j \phi_{ij}$ and ϕ'_j is the relevant k-face of $\sum_i \phi_{ij}$.*

Consequently, if a k-face of a tame m-chain is the sum of a sequence of measurable invariant Radon k-chains, these can be written as the k-faces of tame m-chains that sum to the given m-chain.

Proof Let $\tilde{\phi}_i$ be the relevant face of ϕ_i, and by further decomposing the ϕ_i one may assume $\tilde{\phi}_i$ are Radon. Apply Lemma 4.2.5 to the sequences $\tilde{\phi}_i$ and ϕ'_j, which have the same sum, to obtain measurable invariant Radon k-chains ϕ'_{ij} s.t. $\phi'_j = \sum_i \phi'_{ij}$, $\tilde{\phi}_i = \sum_j \phi'_{ij}$.

We have to write the ϕ'_{ij} as k-faces (i.e. projections) of measurable invariant Radon m-chains ϕ_{ij} s.t. $\phi_i = \sum_j \phi_{ij}$. We do this as follows (fix i, j): ϕ'_{ij} is dominated by $\tilde{\phi}_i$. Take the Radon-Nikodym derivative r_{ij} of ϕ'_{ij} w.r.t. $\tilde{\phi}_i$, which is a function on G^{k+1}, expand it to a function on G^{m+1} by composing it with the relevant projection, and multiply it by $d\phi_i$ to obtain ϕ_{ij}.

The only thing that needs further proof is that ϕ_{ij} is measurable. Denote by x' the variable in G^{k+1} and by (x, x') the variable in G^{m+1}, $(x, x') \mapsto x'$ being the relevant projection (i.e. the relevant k-face of the m-simplex (x, x') is the k-simplex x'). We have to prove that applying ϕ_{ij} to a fixed test-function $h(x, x')$ is measurable in ω. Since every test-function is a non-decreasing limit of finite positive combinations of functions

of the form $f(x)g(x')$, f, g test-functions (Rmk. 2.2.1), we may assume $h(x, x') = f(x)g(x')$ is of that form. Now, for such test-function, applying ϕ_{ij} to it is the same as applying ϕ_i and replacing g by g multiplied by the Radon-Nikodym derivative r_{ij}. Thus if we keep f fixed and concentrate on the dependence on g, then the transition from ϕ_i applied to $f(x)g(x')$ to ϕ_{ij} applied to the same is given by the lattice operations in the proof of Lemma 4.2.5, which preserve measurability by Lemma 4.2.3.

QED

Theorem 4.2.7 *Let a 2nd-countable locally compact group act in a Borel manner on a standard Borel space. Then if two (Borel- resp. universally-measurably) tame* 1*-chains (Def. 4.2.1) have a common vertex, then they are sides of the same (Borel- resp. universally-measurably) tame* 2*-chain.*

Consequently, the relation between (Borel- resp. universally-measurable) tame 0*-chains to be (Borel- resp. universally-measurably) tamely equidecomposable is transitive.*

Proof Decomposing the (tame) common vertex into a sum of measurable invariant Radon 0-chains and applying Lemma 4.2.6, we may assume all the chains are (measurable invariant) Radon.

Let ϕ_1 (on 1-simplices (x_0, x_1)) and ϕ_2 (on 1-simplices (x_0, x_2)) be the two 1-chains. They have a common vertex ϕ_0 (on 0-simplices (x_0)). For each fixed $\omega \in \Omega$, we construct the 2-chain ϕ_3 as follows: disintegrate the two 1-chain w.r.t. the projections on G defined by the common vertex (about disintegration of measures see [Bo-I]). This gives families $(\nu^{(1)}_{x_0})_{x_0 \in G}$, $(\nu^{(2)}_{x_0})_{x_0 \in G}$ of probability measures on G s.t. for non-negative Borel $f(x_0, x_i)$ $(i = 1, 2)$ $x_0 \mapsto \int_G f(x_0, x_i)\, d\nu^{(i)}_{x_0}(x_i)$ is Borel and

$$\phi_1 = \int_G \delta_{x_0} \otimes \nu^{(1)}_{x_0}\, d\phi_0(x_0), \quad \phi_2 = \int_G \delta_{x_0} \otimes \nu^{(2)}_{x_0}\, d\phi_0(x_0)$$

(the integration of measures is defined, as usual, by applying test-functions). These $\nu^{(1)}_{x_0}$ and $\nu^{(2)}_{x_0}$ are determined by ϕ_1 and ϕ_2 (for fixed ω) up to change of the $\nu^{(i)}$'s in a ϕ_0-null set.

Now define, for each fixed ω:

$$\phi_3 := \int_G \delta_{x_0} \otimes \nu^{(1)}_{x_0} \otimes \nu^{(2)}_{x_0}\, d\phi_0(x_0)$$

as a measure on 2-simplices $(x_0, x_1, x_2) \in G^3$. It is standard to check that this indeed defines a Radon measure, with projections ϕ_1 and ϕ_2 and that the dependence on ω defines an invariant 2-chain. It remains to prove that the chain ϕ_3 is measurable.

As in the proof of Lemma 4.2.6, it suffices to check test-functions of the form $h_0(x_0)h_1(x_1)h_2(x_2)$ where the h_i are test-functions. Fix h_1 and h_2 and perform the integration on the $\nu^{(i)}$'s. Then it is clear that, for the dependence on h_0, we have the situation of Lemma 4.2.4 for $\langle \phi_0, h_0 \rangle$, $\langle \phi_1, h_0 \otimes h_1 \rangle$, $\langle \phi_2, h_0 \otimes h_2 \rangle$ and $\langle \phi_3, h_0 \otimes h_1 \otimes h_2 \rangle$. Therefore applying that lemma gives the measurability of ϕ_3.

QED

Definition 4.2.8 *Let* Λ *be some right-invariant measure on* G*. A function* $f : \Omega \to \overline{\mathbf{R}^+}$ *is called (Borel- resp. universally- (measurably))* $\frac{d\Lambda}{d}$**-tame** *if* $\forall \omega \in \Omega$ $\mathcal{O}_\omega f$ *is* Λ*-measurable, and the* 0*-chain enhancement* $f\frac{d\Lambda}{d}$ *is tame (Def. 4.2.1).* $\frac{d\Lambda}{d}$**-Radon** *functions are defined analogously.*

Every non-negative Borel function is Borel-measurably $\frac{d\lambda}{d}$-tame for λ a Haar measure.

We also have: if f is a Borel function s.t. the set $\{x \in G : \mathcal{O}_\omega f(x) \neq 0\}$ has countable closure for each $\omega \in \Omega$, then f is Borel-measurably $\frac{d\mathrm{count}}{d}$-tame. This follows from the following "invariant" way to decompose a countable close set K in G to countably many discrete sets (on which, of course, the counting measure is Radon): choose a right-invariant metric in G. Decompose the set of isolated points $x \in K$ to the countably many discrete sets $\{x \in K : 2^{-(k+1)} < \mathrm{distance}(x, K \setminus \{x\}) \leq 2^{-k}\}$. Then do the same to all derivatives of K. (Note that there is a countable ordinal γ s.t. the γ-derivative of the (closed) support of $\mathcal{O}_\omega f$ is $\emptyset$ for every $\omega \in \Omega$. This follows from Prop. 2.7.1 and from the considerations in the footnote in §2.7).

Also, let an invariant probability measure μ be given in Ω (and λ a Haar measure on G). Then any $\frac{d\Lambda}{d\lambda}\mu$-integrable (Borel) function f can be changed on a Borel $\frac{d\Lambda}{d\lambda}\mu$-null set to become a Borel-measurably $\frac{d\Lambda}{d}$-Radon (Borel) functionf'.

To see this, note that, in the context of §2.4, $f \in \mathrm{Pre}$, thus the 0-chain $f\frac{d\Lambda}{d}$ is measurable and has a Radon expectation. This implies that for any test-function h on G, the set

$$\Omega_h := \{\omega \in \Omega : \int_G f(x\omega)h(x)\, d\Lambda(x) < \infty\}$$

is (μ-)conull.

Choose a countable set $\mathcal{F}$ of test-functions on G s.t. every test-function h is majorized by some $h' \in \mathcal{F}$. Then the intersection Ω' of all the Ω_h for all test-functions h is equal to $\cap_{h\in\mathcal{F}}\Omega_h$, hence is ($\mu$-)conull. Moreover, Ω' is G-invariant. At an $\omega \in \Omega'$, $\frac{d\Lambda}{d}f$, i.e. $f(x\omega)\, d\Lambda(x)$, is a Radon measure. Pick a G-invariant *Borel* (μ-)conull $\Omega'' \subset \Omega'$. (Prop. 2.7.6). Replacing f by $f' = f \cdot 1_{\Omega''}$ will do.

4.3 Recovery of the Original Measure from the Infinitesimal Measure

In this § we deal with two related problems mentioned in §4.1: to recover the measure from the infinitesimal measure, and to find a function which, enhanced by one invariant measure, will be equidecomposable to a given function enhanced by another invariant measure.

Our setting is a unimodular 2nd countable locally compact group G (with Haar measure λ) acting in a Borel manner on a standard Borel space Ω.

Occasionally, Ω will be endowed with a (probability) measure μ such that the action of G is measure-preserving, in other words, μ is invariant.

Also, we shall sometimes have a right-invariant measure Λ on G.

Remark 4.3.1 Suppose G_0 is an open subgroup of the acting group G. Consider the restriction $\lambda_0 := \lambda|_{G_0}$ which is a Haar measure in G_0. Then it is clear that for any invariant measure Λ in G, with restriction $\Lambda_0 := \Lambda|_{G_0}$, enhancing a function or measure on Ω by $\frac{d\Lambda}{d\lambda}$ is the same as enhancing it by $\frac{d\Lambda_0}{d\lambda_0}$, referring to G_0 as the acting group. Moreover, any 1-chain for G_0 can be viewed as a 1-chain for G, hence two enhanced functions equidecomposable w.r.t. G_0 are *ipso facto* equidecomposable w.r.t. G.

This implies that solving our above problems for G_0 will solve them for G. That applies, in particular, to G – a Lie group and G_0 – its identity component. Thus we may assume our Lie groups are *connected*.

4.3.1 Reduction to the Counting Measure Case

Note the two formulas in §2.8.2, which are, in fact, instances of equidecomposability of enhanced functions as in CHG, namely:

- Let $f : \Omega \to \overline{\mathbf{R}^+}$ be Borel-measurably $\frac{d\Lambda}{d}$-tame. (Similar consideration will hold for universally-measurably tame functions.)

  Let $h : G \to \overline{\mathbf{R}^+}$ be non-negative Borel.

  Consider the weighted graph $F(x, y; \omega) = h(xy^{-1})f(x\omega)$ and the 1-simplex of measures (Λ, λ). That defines a Borel-measurably tame 1-chain (consider $(x, y) \mapsto h(xy^{-1})$ as a test-function on G^2, and approach it by sums of test-functions with separated variables x and y), thus its source and target are equidecomposable. This means that

$$\left[\int_G h(x^{-1})\, d\lambda(x)\right] f(\omega)\frac{d\Lambda}{d\lambda}$$

  is Borel-tamely equidecomposable to the function

$$\omega \mapsto \int_G h(x)f(x\omega)\, d\Lambda(x).$$

  The latter function is Borel (consider h as a test-function).

Consequently, every Borel-measurably $\frac{d\Lambda}{d}$-tame function $f:\Omega\to\overline{\mathbf{R}^+}$ is, enhanced by $\frac{d\Lambda}{d\lambda}$, Borel-tamely equidecomposable with a Borel function.

If one takes $\Lambda=\lambda$, one finds that any non-negative Borel function is Borel-tamely equidecomposable to a function which is a "convolution" of a Borel function on Ω and an $\mathrm{Ł}^1$-function on G, hence is l.s.c. (lower semi-continuous) for some Lusin topology on Ω (see §A.3).

- Let $f:\Omega\to\overline{\mathbf{R}^+}$ be Borel-measurably $\frac{d\mathrm{count}}{d}$-tame.

  Let $h:G\to\overline{\mathbf{R}^+}$ be Borel s.t. $h(x^{-1})$ is Λ-integrable.

  Consider the weighted graph $f(x\omega)h(xy^{-1})$ and the 1-simplex of measures $(\mathrm{count}_G,\Lambda)$. Note the discussion of this graph in §2.8.2. It is noted there that the σ-finiteness requirements of CHG hold for all ω, and that discussion shows that the corresponding 1-chain is Borel-tame.

  Thus we have
  $$\left[\int_G h(x^{-1})\,d\Lambda(x)\right]f(\omega)\frac{d\mathrm{count}}{d\lambda}$$
  Borel-tamely equidecomposable with
  $$g(\omega)\frac{d\Lambda}{d\lambda}\quad\text{for } g(\omega):=\int_G f(x\omega)h(x)\,d\mathrm{count}_G(x).$$

  g is Borel (consider h as a test-function).

  We conclude that for any invariant measure Λ that is not $\{0,\infty\}$-valued (thus $\exists$ a Λ-integrable $h\ge 0$ with $\int h\,d\Lambda\neq 0$), and for any non-negative Borel-measurably $\frac{d\mathrm{count}}{d}$-tame function f on Ω, f, enhanced by $\frac{d\mathrm{count}}{d\lambda}$, is Borel-tamely equidecomposable to some Borel function enhanced by $\frac{d\Lambda}{d\lambda}$.

  Therefore for any invariant μ, $\frac{d\mathrm{count}}{d\lambda}\mu$ can be reconstructed from $\frac{d\Lambda}{d\lambda}\mu$ (Recall that any $\frac{d\mathrm{count}}{d\lambda}\mu$-integrable Borel function is equal, outside a Borel $\frac{d\mathrm{count}}{d\lambda}\mu$-null set, to a Borel-measurably $\frac{d\mathrm{count}}{d}$-tame Borel function. – see the end of §4.2)

To conclude, if one can recover μ from $\frac{d\mathrm{count}}{d\lambda}\mu$ one can recover μ from any $\frac{d\Lambda}{d\lambda}\mu$, if Λ is not $\{0,\infty\}$-valued; Borel-tame equidecomposability allows us to pass from functions enhanced by such Λ to usual functions (even to functions l.s.c. for a suitable Lusin topology) and from functions enhanced by count_G to functions enhanced by such Λ.

So, it remains to try to recover μ from $\frac{d\mathrm{count}}{d\lambda}\mu$, and to try to find functions which, enhanced by $\frac{d\mathrm{count}}{d\lambda}$ will be equidecomposable with given ordinary functions, which may be assumed l.s.c. for some Lusin topology.

4.3.2 Finding E with Discrete $\mathcal{O}_\omega E$ and $\mathit{satur}\,E$ the Whole Space – Analogy to Ambrose-Kakutani

From now on we restrict ourself to G a unimodular *connected Lie group.*

As noted in Rmk. 2.8.6, in the case of $\mathbf{R}$-action (i.e. a flow) one may use the method of Ambrose and Kakutani [AK] (see also [J], [Na]), to recover the original measure from the infinitesimal measure (in fact, from the restriction of the infinitesimal measure to suitable $E\subset\Omega$ s.t. the original system has the structure of a "flow under a function"). This can be pursued for more general groups G rather than $\mathbf{R}$, and one may try not to refer to a particular invariant measure μ, thus speaking about G-Borel spaces. This is done in [Ke] and [FHM] (cf. also [Wg]). Our approach will be differential-geometric, and seems different from theirs (cf. [Ra]).

Ambrose-kakutani teach us to look for Borel sets $E\subset\Omega$ with $\mathcal{O}_\omega E$ *discrete* in G and $\mathit{satur}\,E$ conull, preferably $\mathit{satur}\,E=\Omega$

One case when such E does not exists is when *the stabilizer of some* $\omega\in\Omega$ (i.e. the subgroup $\{x\in G: x\omega=\omega\}\subset G$) *is a non-discrete closed subgroup (i.e. of positive dimension)* – as in Exm. 2.4.7 items 4 and 5. Note that we may (and do) assume that Ω is a G-compact metric space – see §A.3, thus the stabilizer is always a *closed* subgroup. In such cases one cannot expect to recover μ from $\frac{d\mathrm{count}}{d\lambda}\mu$.

Take for example $G = SO(3)$ (with normalized Haar measure λ) acting on the union of two concentric spheres in R^3 by rotations (compare Exm. 2.4.7 item 5). Taking as μ any non-trivial convex combination of the normalized invariant areas on the spheres, one gets the same $\frac{d\text{count}}{d\lambda}\mu = \infty \cdot \text{count}$.

Note that since the stabilizer of $y\omega$ is a conjugate of the stabilizer of ω, the set of all ω with non-discrete stabilizer is invariant. Moreover, in the case that Ω is a G-compact space this set is *closed* (see [Ra]). Indeed, it is the projection on Ω of the compact set

$$\{(\omega, v) \in \Omega \times T_eG : \|v\| = 1,\ \forall 0 \le t \le 1\ \exp(tv)\omega = \omega\}$$

where some norm on the tangent space T_eG is understood.

But what if the stabilizers are discrete? we have the following analog of Ambrose-Kakutani:

Theorem 4.3.2 *(see [Ke] and [FHM]) Let G be a Lie group acting in a Borel manner on a standard Borel space Ω.*

Partition Ω into two invariant Borel sets: the set Ω_1 of $\omega \in \Omega$ with non-discrete stabilizer and its complement – the set Ω_0 of $\omega \in \Omega$ with discrete stabilizer. Then the latter set contains a Borel $E \subset \Omega$ with $\mathcal{O}_\omega E$ discrete for all $\omega \in \Omega$, and with $\mathit{satur}\, E = \Omega_0$.

Proof W.l.o.g. assume Ω is a G-compact metric space (see §A.3).

Let n be the dimension of the Lie group G, and choose a right Haar measure λ in G.

In the set of all closed subsets of G or of all open subsets of G we take the Effros Borel structure (see §2.7), obtained by identifying each closed $F \subset G$ (resp. each open $U \subset G$) with the set of members u of a fixed countable open base to the topology that satisfy $u \cap F = \emptyset$ (resp. $u \subset U$), this set being a member of $2^{\mathbf{N}}$.

Our strategy will be to correspond to each $\omega \in \Omega_0$ a non-empty discrete set $F(\omega) \subset G$ in a Borel and *equivariant* manner, the latter meaning that $F(\omega) = F(a\omega)a$, $a \in G$. Then the set $E = \{\omega : e \in F(\omega)\}$ has the property

$$a\omega \in E \Leftrightarrow e \in F(a\omega) \Leftrightarrow a \in F(\omega), \quad a \in G$$

Thus $\mathcal{O}_\omega E$ is $F(\omega)$, which is discrete non-empty. In particular, $\mathit{satur}\, E = \Omega_0$.

Call a function $f : \Omega \to \mathbf{R}$ $\mathcal{C}^\infty$ if f is continuous, $\forall\omega \in \Omega$ $\mathcal{O}_\omega f$ is $\mathcal{C}^\infty$ on G and $\omega \mapsto \mathcal{O}_\omega f$ is continuous from Ω to the Fréchet space $\mathcal{C}^\infty(G)$ with the usual topology of uniform convergence on compacta of all partial derivatives. For any continuous $f : \omega \to \mathbf{R}$ and any $\mathcal{C}^\infty$-function with compact support $h : G \to \mathbf{R}$, the convolution

$$\omega \mapsto \int_G f(x\omega)h(x)$$

is $\mathcal{C}^\infty$. Thus any continuous function on Ω can be uniformly approximated by $\mathcal{C}^\infty$ functions.

We wish to be able to bring together differential geometric notions pertaining to different points in G. We do this by equating the tangent spaces via *right translations*. Namely, choose a basis to the tangent space T_eG, and "expand" it to n right-invariant vector fields X_i, $i = 1, \ldots, n$ forming a basis to the tangent space at each point. We use the notation Xf, where f is a $\mathcal{C}^\infty$ function on G and X is a tangent vector or vector field on G, understood as acting on f as usual (see [Hi]).

For any $\alpha \in \mathbf{R}^n$ there is a unique differential form (which is $\mathcal{C}^\infty$) $\tilde{\alpha}$ on G satisfying $\langle\tilde{\alpha}, X_i\rangle = \alpha_i$, $i = 1, \ldots, n$ These $\tilde{\alpha}$ are the **right Maurer-Cartan forms on** G (see [Co] §4.4).

Let $f : \Omega \to \mathbf{R}$ be $\mathcal{C}^\infty$, let $\alpha \in \mathbf{R}^n$ and consider the differential form on G, depending on $\omega \in \Omega$, $d\mathcal{O}_\omega f - \tilde{\alpha}$. (Note that this, as a member of the Fréchet space of the differential forms on G, depends continuously on (ω, α), thus there is no question about Borelness of the subsets of Ω to be considered below.)

Suppose we find an *invariant* (i.e. constant on orbits) Borel function $\omega \in E' \mapsto \alpha(\omega) \in \mathbf{R}^n$ where E' is a Borel invariant set $E' \subset \Omega$, s.t. $\forall\omega \in E'$ the (closed) set $F_0(\omega) \subset G$ consisting of the points where $d\mathcal{O}_\omega f - \widetilde{\alpha(\omega)}$ vanishes (i.e. gives the zero element of the cotangent space) contains some isolated points. Then the mapping $\omega \in E' \mapsto F_0(\omega)$ is Borel, and the right-invariant way in which $\tilde{\alpha}$ was constructed (and the invariance of $\omega \mapsto \alpha(\omega)$) implies $F_0(\omega) = F_0(a\omega)a$ $a \in G$. To get a discrete non-empty set $F(\omega)$ out of $F_0(\omega)$ in an equivariant way, just take:

$$F(\omega) := \left\{x \in G : \text{distance}\,(x, F_0(\omega) \setminus \{x\}) > \frac{1}{2}\min\left(1, \sup_{y \in F_0(\omega)} \text{distance}\,(y, F_0(\omega) \setminus \{y\})\right)\right\}$$

Hence we shall be done if we can cover Ω with a countable union of invariant Borel sets E' with invariant (i.e. constant on orbits) Borel function $\omega \in E' \mapsto \alpha(\omega) \in \mathbf{R}^n$ s.t. $\forall \omega \in E'$ the set where $d\mathcal{O}_\omega f - \widetilde{\alpha(\omega)}$ vanishes contains some isolated point (reduce this covering to a disjoint countable covering and take the union of the corresponding E).

Using the basis X_i, the differential form $d\mathcal{O}_\omega f - \tilde{\alpha}$ is described as a mapping $G \to \mathbf{R}^n$, namely

$$x \mapsto \left(\langle d\mathcal{O}_\omega f - \tilde{\alpha}, X_i\rangle\right)_i = \left(X_i \mathcal{O}_\omega f - \alpha_i\right)_i \tag{37}$$

This mapping has an invertible differential at some point of G iff its Jacobian matrix, which is, in fact, the Hessian of f

$$H_{ij}(x;\omega;f) = \left(X_j(X_i \mathcal{O}_\omega f)\right)_{ij} \quad x \in G, \omega \in \Omega \tag{38}$$

is non-singular at this point. If, for some ω and α, this Jacobian is non-singular at some $a \in G$ where $d\mathcal{O}_\omega f - \tilde{\alpha}$ vanishes, then at every such point a (37) is locally 1-1, by the inverse function theorem, hence every such a is an isolated zero.

Take as our $E' = E'(f)$ above the set of all ω with $H_{ij}(x;\omega;f)$ (38) non-singular somewhere on G. E' is open, its complement being the closed set of the ω with H_{ij} singular everywhere in G.

In order to carry out our plan, we have to correspond to every $\omega \in E'$, in a Borel and invariant (i.e. constant on orbits) manner, an $\alpha \in \mathbf{R}^n$ s.t. $d\mathcal{O}_\omega f - \tilde{\alpha}$ vanishes somewhere where H_{ij} is non-singular. This means that α belongs to the image $I(\omega; f) \subset \mathbf{R}^n$ by

$$x \mapsto D(x) := \left(X_i \mathcal{O}_\omega f\right)_i$$

of the set of $x \in G$ where $H_{ij}(x)$ is non-singular.

But by the way H_{ij} and $D(x)$ were defined, replacing ω by $a\omega$ ($a \in G$) would change $H_{ij}(x)$ into $H_{ij}(xa)$ and $D(x)$ into $D(xa)$. Therefore $\omega \mapsto I(\omega; f)$ is invariant (i.e. constant on orbits). $I(\omega; f)$ is open in G, and depends on ω in a Borel manner. Hence we can take as $\alpha(\omega)$ the first element belonging to $I(\omega; f)$ in a fixed dense sequence in $\mathbf{R}^n$.

Thus we can fulfill the requirements of the theorem for $E' = E'(f)$, that is, find a Borel E with discrete $\mathcal{O}_\omega E$ and *satur* $E = E'$. Since Ω is compact metric, the union of the open $E'(f)$ for all $\mathcal{C}^\infty$-functions f is covered by a countable number of them, so the requirements of the theorem hold also for this union. To conclude, we prove that this union is all of Ω_0 – the set of ω with discrete stabilizer. (Certainly the union is contained in Ω_0, since an E as above cannot exist for $\omega \notin \Omega_0$.)

So suppose ω belongs to the complement of $E'(f)$ for all $\mathcal{C}^\infty$-functions f. By the above, this means that for any such f, the Hessian $H_{ij}(x;\omega;f)$ (38) is singular $\forall x \in G$.

Let $\mathcal{M}$ be the set of matrix values attained by $\left(H_{ij}(e;\omega;f)\right)_{ij}$ at the unity e, *for all $\mathcal{C}^\infty$-functions f on* Ω. $\mathcal{M}$ is a vector subspace of the vector space of $n \times n$ matrices. Since f may be replaced by $f(a\omega)$ and H_{ij} is defined in a right-equivariant manner, $\mathcal{M}$ is also the set of matrix values obtained by $H_{ij}(x;\omega;f)$ for any fixed x.

Now the Hessian H_{ij} (38) is symmetric, since (see [Hi]):

$$H_{ij} - H_{ji} = X_j(X_i \mathcal{O}_\omega f) - X_i(X_j \mathcal{O}_\omega f) = X_j \langle d\mathcal{O}_\omega f, X_i\rangle - X_i \langle d\mathcal{O}_\omega f, X_j\rangle = \langle dd\mathcal{O}_\omega f, [X_j, X_i]\rangle = 0$$

In this situation, one has:

Lemma 4.3.3 *([Fl] Lemma 1.) Let $\mathcal{M}$ be a linear subspace of the space of all $n \times n$ symmetric matrices over an infinite field $\mathbf{F}$ with characteristic $\neq 2$, with all members of $\mathcal{M}$ singular (i.e. having zero determinant). Then there exists a non-zero vector $v \in \mathbf{F}^n$ with $v^T M v = 0$ $\forall M \in \mathcal{M}$.*

Proof Let r be the maximum rank of members of $\mathcal{M}$. By the assumption $r \leq n-1$. Let $A \in \mathcal{M}$ be of rank r. Replacing, if necessary, every member M of $\mathcal{M}$ by $Z^T M Z$, where Z is a fixed non-singular matrix, we may assume that A is diagonal, with first r diagonal entries $\neq 0$ and the others 0. We prove that

$$\forall M \in \mathcal{M}\ \forall i > r, j > r\ M_{ij} = 0$$

Indeed, consider the linear family $M + tA$ of members of $\mathcal{M}$, hence having rank $\leq r$. Consider in them the $(r+1)$-minor built from the first r rows and columns, the i-th row and the j-th column. Its determinant, which is an r -th degree polynomial in t, is 0 for every t. So its coefficient of t^r, which is $A_{11}\cdots A_{rr}M_{ij} = 0$, implying $M_{ij} = 0$.

In particular, $M_{nn} = 0\ \forall M \in \mathcal{M}$, and the conclusion of the lemma holds with $v = (0,\ldots,0,1)$.
QED

So, by Lemma 4.3.3, $\exists$ a $v \in \mathbf{R}^n$, v not the zero vector, s.t. for our ω, $\forall$ $\mathcal{C}^\infty$-function $f : \Omega \to \mathbf{R}$,

$$\sum_{ij} H_{ij}(x;\omega;f)v_i v_j = 0$$

for all $x \in G$. By (38) this means:

$$(\sum v_j X_j)\left[(\sum v_i X_i)\mathcal{O}_\omega f\right] = 0 \quad x \in G, \omega \in \Omega \tag{39}$$

This implies that if $t \mapsto x(t) = \exp(t\sum v_i X_i)$ is the integral curve of the right-invariant non-zero vector field $\sum v_i X_i$ tracing the one-parameter subgroup corresponding to this vector, then $\mathcal{O}_\omega f(x(t))$ has zero second derivative, hence is linear. Since f is bounded (continuous on a compact) $\mathcal{O}_\omega f(x)$, i.e. $f(x\omega)$, is constant on that one-parameter subgroup. But we have noted that $\mathcal{C}^\infty$-functions are dense in $\mathcal{C}(\Omega)$. One concludes that $x\omega = \omega$ for every x in the above one-parameter subgroup, i.e. the stabilizer of ω has positive dimension, thus $\omega \notin \Omega_0$.

This concludes the proof.
QED

4.3.3 Using E with $\mathcal{O}_\omega E$ Non-Empty Countable Closed

Having Thm. 4.3.2 at hand, one can address the problem of recovery of μ from $\frac{d\text{count}}{d\lambda}\mu$ and that of finding a function which, enhanced by $\frac{d\text{count}}{d\lambda}$ will be equidecomposable to a given usual function (which would solve the former problem).

Thus assume G is a unimodular connected Lie group, and that $E \subset \Omega$ is Borel s.t. for all $\omega \in \Omega$ $\mathcal{O}_\omega E$ is a *non-empty countable closed* subset of G.

We will use the "nearest point" construction of §2.8.4, in particular Prop. 2.8.10 (iv). One may note that, at least for the recovery of μ, we can avoid this construction by applying CHG, instead of to the graph (31) in §2.8.4 as below, to the Graph $\{(x,y) \in G^2 : y \in \mathcal{O}_\omega E\}$. Here one uses the fact that Ω is a countable-to one Borel image of $E \times G$ by $(\omega \in E, x \in G) \mapsto x\omega$.

We use the same notations as in §2.8.4. In particular, for $\omega \in \Omega$, $\pi(\omega) = \pi_E(\omega) \in G$ is the unique nearest point to 0 in $\mathcal{O}_\omega E$, if it exists (if there is no unique nearest point, $\pi_E(\omega)$ is undefined); for $\omega \in E$, $P(\omega) = P_E(\omega)$ is the set of $x \in G$ s.t. 0 is the unique nearest point to x in $\mathcal{O}_\omega E$.

By §2.8.4, the mapping $(\omega, x) \mapsto x\omega$ maps the set

$$\{(\omega,x) \in E \times G : x \in P_E(\omega)\}$$

in a (Borel) 1-1 manner onto the set

$$\Omega' = \{\omega \in \Omega : \pi_E(\omega) \text{ is defined}\}$$

i.e.

$$\Omega' = \{\omega \in \Omega : \text{ there is a unique nearest point to 0 in } \mathcal{O}_\omega E\}.$$

Being a Borel 1-1 image of a Borel set, Ω' is Borel. Also, Ω' has the property that $\mathcal{O}_\omega \Omega'$ is (Haar-)conull for all $\omega \in \Omega$ (Prop. 2.8.8 in §2.8.4). Thus, as far as a Borel function on Ω is considered as a chain ("enhanced" by $\frac{d\lambda}{d\lambda}$) it does not matter if we restrict it to Ω'.

Also, this 1-1 mapping $(\omega, x) \mapsto x\omega$ is a Borel isomorphism, the inverse mapping $\Omega' \to \{(\omega, x) \in E \times G : x \in P_E(\omega)\}$ being:

$$\omega \mapsto \left(\pi(\omega)\omega, \pi(\omega)^{-1}\right).$$

Let $f : \Omega \to \overline{\mathbf{R}^+}$ and $s : G \to \overline{\mathbf{R}^+}$ be Borel. Consider the graph used to prove Prop. 2.8.10 (iii) with $s \equiv 1$, namely, the graph

$$\{(x, y) \in G^2 : y \text{ is the unique nearest point to } x \text{ in } \mathcal{O}_\omega E\}$$

weighted by $f(x\omega)$, with the simplex of measures $(\lambda, \mathrm{count}_G)$. As we have seen in §2.8.4, the obtained 1-chain is Borel-measurable. Moreover, it is Borel-tame – see the discussion following Def. 4.2.8. The vertices, being hence Borel-tamely equidecomposable, are:

Source: the ordinary Borel function (restricted to Ω', which, by the above, does not matter):

$$\omega \mapsto f(\omega)$$

Target: the Borel function supported on E, enhanced by $\frac{d\mathrm{count}}{d\lambda}$:

$$\left[\omega \mapsto \int_{P_E(\omega)} f(x\omega)\, d\lambda(x)\right] \frac{d\mathrm{count}}{d\lambda} \tag{40}$$

Thus we have:

Theorem 4.3.4 *Let a unimodular connected Lie group G with Haar measure λ act in a Borel manner on a standard Borel space Ω. Let $E \subset \Omega$ be Borel with non-empty countable closed $\mathcal{O}_\omega E$ for every ω. Then any non-negative Borel function f on Ω is Borel-tamely equidecomposable (Def. 4.2.1) with a non-negative Borel function on E enhanced by $\frac{d\mathrm{count}}{d\lambda}$, namely, with the enhanced function given by (40).*

Consequently, if such an E exists, then any invariant probability measure on Ω can be recovered from the corresponding $\frac{d\mathrm{count}}{d\lambda}$-enhanced infinitesimal measure, even from its restriction to E.

QED

Remark 4.3.5 The "flow under a function" scene, as well as its generalization in this §, can be viewed as follows:

We have a standard space Ω acted in a Borel manner by a unimodular 2nd-countable locally compact group G with Haar measure λ. We have a Borel subset E and we pick a right-invariant measure Λ on G. On E we are given a measure ν, which should hopefully be the restriction to E of $\frac{d\Lambda}{d\lambda}\mu$ for some G-invariant (say, probability) measure μ on Ω.

Thus in the "flow under a function" construction we start with a standard measure space (E, ν) with $\mathbf{Z}$-action and a given positive measurable function, we embed it into a larger space Ω with $\mathbf{R}$-action s.t. the given $\mathbf{Z}$-action is given by T_E and the given function is ρ_E (as defined for the $\mathbf{R}$-case in §2.8), and one finds a μ, invariant under the $\mathbf{R}$-action, s.t. the initial measure ν in E is the restriction to E of $\frac{d\mathrm{count}}{dt}\mu$.

For the general case of an $E \subset$ an Ω acted by G and a ν on E, we have at our disposal the relation of two functions on E being tamely equidecomposable when enhanced by $\frac{d\Lambda}{d}$. A necessary condition for the existence of a μ is that ν gives the same integral to such functions, which we can view as a substitute for the property of G-invariance for a ν "known" only on E. If this condition is satisfied, and if moreover we have the counterpart of Thm. 4.3.4 – every ordinary Borel function on Ω is tamely equidecomposable with some function supported on E, enhanced by $\frac{d\Lambda}{d\lambda}$, then we know the integral w.r.t. the sought-for μ of any Borel function and since this proposed integral is countably additive on functions, we have a unique μ, where we have to check that this μ is σ-finite (say, by showing that 1_Ω is a countable sum of nonnegative functions tamely equidecomposable with ν-integrable nonnegatve functions on E). This view is of interest even when $\Lambda = \lambda$, so we are given an $E \subset \Omega$ and a μ on E and seek a G-invariant μ s.t. $\mu|_E = \nu$.

In the case of the "flow under a function", ν is T_E-invariant, and it is easy to show that functions on E are equidecomposable when the $\mathbf{R}$-action on Ω is considered and one enhances the functions by $\frac{d\mathrm{count}}{d\lambda}$ iff they are equidecomposable w.r.t. the discrete $\mathbf{Z}$-action given by T_E, so ν gives the same integral to such functions.

Since, moreover, any Borel function f on Ω is equidecomposable to a function on E enhanced by $\frac{d\mathrm{count}}{dt}$, given, say, by (29) for $s \equiv 1$:

$$\left[\int_0^{\rho(\omega)} f(T^t\omega)\,dt\right] \frac{d\mathrm{count}}{dt},$$

we are sure the required μ on Ω exists and get a formula for $\int f\,d\mu$ in terms of ν:

$$\int_\Omega f(\omega)\,d\mu(\omega) = \int_E \left[\int_0^{\rho(\omega)} f(T^t\omega)\,dt\right] d\nu(\omega).$$

In the case dealt with in this § – G is a Lie group and E is such that $\mathcal{O}_\omega E$ is always countable closed, one can speak of a groupoid structure in E instead of the nonexistent T_E, and functions on E are equidecomposable when enhanced by $\frac{d\mathrm{count}}{d\lambda}$ iff they are equidecomposable w.r.t. to the (discrete) groupoid.

5 Some Applications

5.1 Comparison with a Result of G. Helmberg; Persistence and Interruption of Patterns

The continuous Kac Thm. 2.8.1 bears some resemblance to a limit theorem of G. Helmberg [He] This theorem is formulated for an $\mathbf{R}^+$-action. Let us state it, partly using our notation:

Helmberg defines:

$$\begin{aligned} \text{for } t>0 \quad E_t &:= \{\omega \in \Omega : \exists r, s \text{ s.t. } 0 \le r < s \le t \ \&\ r \in \mathcal{O}_\omega E \ \&\ s \notin \mathcal{O}_\omega E\} && (41)\\ r_E(\omega) &:= \inf\{s > 0 : s \in \mathcal{O}_\omega E \ \&\ \exists r \text{ s.t. } 0 \le r < s \ \&\ r \notin \mathcal{O}_\omega E\} && (42)\\ \text{for } t>0 \quad E(t) &:= E_t \cap \{\omega : r_E(\omega) < t\} && (43) \end{aligned}$$

He proves, using discrete approximation (i.e. the $\mathbf{Z}^+$-actions T^{nt} for $t > 0$) the following

Theorem 5.1.1 *(G. Helmberg) Suppose $\left(\Omega, \mathcal{B}, \mu, (T^t)_{t\in\mathbf{R}^+}\right)$ is a measure-preserving flow, and $E \in \mathcal{B}$ is s.t. E_t , $t > 0$ and r_E are measurable, and s.t. satur$E$ is conull. Suppose*

$$\lim_{s\to 0} \mu(E_s) = 0 \tag{44}$$

$$\lim_{s\to 0} \frac{1}{s}\mu(E(s)) = 0 \tag{45}$$

Then

$$\lim_{s\to 0} \frac{1}{s} \int_{E_s} r_E(\omega)\, d\mu(\omega) = 1 - \mu(E) \tag{46}$$

(we have added the inessential requirement that satur$E$ is conull – satur$E$ being always measurable if the E_t's are).[8]

This theorem, for Borel $\mathbf{R}$-action and E Borel, can be proved using the ideas of Section 2. This will be presented here.

Let us remark first, that Helmberg's definitions may be described in the context of "persistence of a certain pattern". Suppose we wish to speak about "moments of exit from E". To do this we may consider the following "pattern": an interval of Time being the union of an interval of stay outside E and a subsequent interval of stay in E. More precisely: for any $A \subset \mathbf{R}$, consider the union of all open intervals $I \subset \mathbf{R}$ such that in the decomposition $I = (I \cap A) \cup (I \cap A^c)$ both intersections are intervals (possibly empty) and $I \cap A^c$ precedes $I \cap A$, this being our "pattern". Let $\Gamma(A)$ be the *complement* of the union of all such intervals, thus $\Gamma(A)$ is composed of Time moments when the pattern is interrupted. For $A = \mathcal{O}_\omega E$ These time moments may be viewed naturally as "moments of exit" from E (for a fixed ω). Similarly, the pattern leading to "moments of entry" will be obtained by requiring $I \cap A$ to precede $I \cap A^c$.

Assume we are in our usual setting of Borel $\mathbf{R}$-action on a standard space and E is Borel.

Note that it is not hard to convince oneself, using Prop. 2.7.1 (i), that for Γ referring to "moments of exit" or to "moments of entry", the closed set $\Gamma(\mathcal{O}_\omega E)$ is a measurable function of ω, being a Borel function of $\overline{\mathcal{O}_\omega E}$.

The graph used in the proof of the continuous Kac Thm. 2.8.1 can be constructed from the the closed set $C = \Gamma(\mathcal{O}_\omega E)$ instead of $\overline{\mathcal{O}_\omega E}$, the latter being the case where the pattern is: "staying out of A". "Return time" and "arrival time" can be defined for general Γ, for example, the arrival time for the pattern of "exit from E" will have the meaning of "the waiting time to exit E". An analog of Thm. 2.8.1 can be proved.

Consequently, Helmberg's $r_E(\omega)$ is a.e. "the waiting time to enter E" and the measurability of $\Gamma(\mathcal{O}_\omega E)$ as a function of ω (for the "entry" pattern) implies Helmberg's condition that $r_E(\omega)$ be measurable.

[8]In the second part of [He] Helmberg defines, for E closed in a compact metric Ω and continuous action, the notions of *ex*E and *in*E (different from ours), these being subsets of Ω. He formulates requirements on them in this topological setting that insure (44) and (45) hence (46). *ex*E is the set of ω in E which will visit E^c in every $[0, t]$-future. Similarly for *in*E.

Also, for a.a. ω, ω belongs to E_t iff in the $[0,t]$-future there is a "moment of exit"; and it belongs to $E(t)$ iff this future contains both a moment of exit and a moment of entry.

Note that (46) involves both the "entry" pattern (r_E) and the "exit" pattern (E_t).

The following lemma is in the spirit of Satz 3 in [He]:

Lemma *For Borel action and E borel, and assuming Helmberg's condition (45), let ρ be a (nonnegative) measurable (ω-dependent) invariant 0-chain with Radon (i.e. Haar) expectation. Then the integral $I(t)$ over $\omega \in E(t)$ of the integral of $1_{]0,t[}$ over ρ is $o(t)$ as $t \to 0$.*

Proof (see the proof of Satz 3 in [He]): Since $E(t)$ depends increasingly on t, $I(t)$ is an increasing function of t. Hence it suffices to prove $I(t/n) = o(1/n)$ as $n \to \infty$ for fixed t, i.e. $nI(t/n) \to 0$ as $n \to \infty$. To compute $I(t/n)$ we integrate $1_{]0,t/n[}$ over ρ, and then integrate over the ω with entry and exit (w.r.t. E) in the $]0, t/n[$-future. Substituting $T^{kt/n}\omega$ for ω, we get the same by integrating $1_{]kt/n,(k+1)t/n[}$ over ρ and then integrating over the set of ω with entry and exit in the $]kt/n,(k+1)t/n[$-future, the latter set having measure $\mu(E(t/n))$. Summing this for $k = 0, \ldots, n-1$, we have $nI(t/n) \leq$ the integral of $\int 1_{]0,t[}\, d\rho$ over a set Ω_1 of ω's with measure $\leq n\mu(E(t/n))$. By (45), $\mu(\Omega_1) \to 0$. Since ρ is measurable with Haar expectation, $\int 1_{]0,t[}\, d\rho$ belongs to $L^1(\Omega)$. Therefore its integral over Ω_1 tends to 0 as $n \to \infty$, and we are done.

QED

Now consider the following ω dependent graph: Let Γ refer to the "exit" pattern. an arrow (t_0, t_1) belongs to the graph iff $t_1 < t_0$, $t_0 \notin \mathcal{O}_\omega E$, $t_1 \in \Gamma(\mathcal{O}_\omega E)$ and $]t_1, t_0] \cap \mathcal{O}_\omega E = \emptyset$. Take the 1-simplex of invariant measures $(dt, \mathrm{count}_{\mathbf{R}})$, and apply CHG (Check that its requirements are satisfied). One finds:

Let

$$E' = \{\omega \in \Omega : 0 \in \Gamma(\mathcal{O}_\omega E) \;\&\; \exists \varepsilon > 0 \text{ s.t. }]0,\varepsilon] \cap \mathcal{O}_\omega E = \emptyset\}$$
$$E'' = \{\omega \in \Omega : \Big] \sup\left(]-\infty, 0] \cap \Gamma(\mathcal{O}_\omega E)\right), 0\Big] \text{ is nonempty and disjoint from } \mathcal{O}_\omega E\}$$

(Note that $\mathcal{O}_\omega E' \subset \Gamma(\mathcal{O}_\omega E)$, $E'' \subset E^c$, and $\mathcal{O}_\omega(E^c \setminus E'') \subset \Gamma(\mathcal{O}_\omega E)$.)

The source: a.e. the ordinary function $1_{E''}$.

The target: $r_E 1_{E'} \frac{d\mathrm{count}}{dt}$.

Thus, by CHG, $r_E 1_{E'}$, enhanced by $\frac{d\mathrm{count}}{dt}$, has expectation $\mu(E'')$.

Consider the 0-chain enhancement $\rho = 1_{E^c \setminus E''} + r_E 1_{E'} \frac{d\mathrm{count}}{d}$ i.e. $1_{\mathcal{O}_\omega(E^c \setminus E'')}\, dt + \mathcal{O}_\omega\left(r_E 1_{E'}\right)\, d\mathrm{count}_{\mathbf{R}}$. By the above, ρ is supported on moments of exit for ω.

To obtain (46), note that, by the above, ρ has expectation $1 - \mu(E)$. Thus $1/t$ times the integral of $1_{]0,t[}$ over ρ has expectation $1 - \mu(E)$. Since ρ is supported on moments of exit, this integral is 0 for $\omega \notin E_t$. For $\omega \in E_t \setminus E(t)$ (i.e. exit but no entry in the $]0,t[$-future, hence only one moment of exit then) the integral differs from $r_E(\omega)$ by no more than t, and one notes (44). The $E(t)$ part is disposed of by the lemma and $r_E(\omega)$ being $\leq t$ there, and we are done.

QED

5.2 Aaronson and Weiss's Kac Functions

In [AW] J. Aaronson and B. Weiss make crucial use of a **Kac function** of a subset $E \subset \Omega$, $\mu(E) > 0$ of a measure space (Ω, μ) on which the group $G = Z^d$ acts measure-preservingly. In Z^d take the l^∞-norm.

They define a *Kac function* as a measurable function $\phi : E \to \mathbf{Z}^+$ satisfying

$$\bigcup_{n=1}^{\infty} \bigcup_{\|x\| \leq n} T^x \{\omega : \phi(\omega) = n\} = \Omega \bmod \mu \tag{47}$$

$$\int_E \phi^d \, d\mu < \infty \tag{48}$$

That is, to every $\omega \in E$ one corresponds a "radius" $\phi(\omega)$ in $G = \mathbf{Z}^d$ s.t. the union, for $x \in E_\omega$, of the cubes with radius $\phi(T^x\omega)$ and center x is a.e. the whole of G, while the "volume" of the cube has finite expectation.

In their above article they seem to promise to prove in a future article that in the ergodic case such a Kac function alway exists, while in the present article they prove its existence (in the ergodic case) in a random sense, i.e. when one passes to an appropriate extension of the dynamical system (in their case, a product) and lets ϕ depend on the points of the extension.

In the sequel, a way to get such a "random" Kac function (for the case *satur* E conull) is presented, which is essentially Aaronson and Weiss's method. Instead of the pointwise ergodic theorem for Z^d-action, which they apply, the HG theorem is used.

Consider the compact group $\mathbf{Z}_2$ of the *dyadic integers*. Every element of $\mathbf{Z}_2$ has an infinite dyadic expansion with ascending powers of 2.

Take the normalized (i.e. probabilistic) Haar measure in $\mathbf{Z}_2^d$, denoted by λ.

One has the $\mathbf{Z}^d$-*dyadic odometer*, which is the set $\mathbf{Z}_2^d$ acted upon measure-preservingly by $\mathbf{Z}^d$ via addition:

$$T^x\alpha = x + \alpha \quad x \in \mathbf{Z}^d, \alpha \in \mathbf{Z}_2^d$$

An element $\alpha \in \mathbf{Z}_2^d$ can be identified with a hierarchy of dyadic partitions of $\mathbf{Z}^d$, where in the n-th step $\mathbf{Z}^d$ is partitioned into cubes with side 2^n (to be called n-cubes of the hierarchy), given by inequalities of the form

$$k_i \le x_i < k_i + 2^n, \quad i = 1, \dots, d$$

In the hierarchy corresponding to an $\alpha \in \mathbf{Z}_2^d$, each n-cube will consist of those $x \in \mathbf{Z}^d$ with common $n+1, n+2, \dots$'th digits in the dyadic expansion of the d coordinated of $x + \alpha$. When the odometer is identified with the set of hierarchies of partitions of $\mathbf{Z}^d$, $\mathbf{Z}^d$ acts on these hierarchies by shift.

Now consider the product dynamical system $\Omega \times \mathbf{Z}_2^d$. Our aim is to prove the existence of a Kac function $\phi(\omega, \alpha)$ on $\Omega \times \mathbf{Z}_2^d$, i.e. ϕ has to satisfy

$$\bigcup_{n=1}^{\infty} \bigcup_{\|x\| \le n} T^x \left\{(\omega, \alpha) : \phi(\omega, \alpha) = n\right\} = \Omega \bmod \mu \times \lambda \tag{49}$$

$$\int_E \phi^d \, d(\mu \times \lambda) < \infty \tag{50}$$

We correspond to every (ω, α) a graph $F(\omega, \alpha) \subset G^2$, in a measurable and invariant way, so that a.e. from every $x \in G$ emanates a unique edge, which terminates in $\mathcal{O}_\omega E$, while the set $S = S(\omega, \alpha)$ of y's from which an edge goes to 0 is a.e. (in E) "thick" in the sense that

$$\mathrm{card}(S) \ge M \cdot (\mathrm{diameter}(S \cup \{0\}))^d \tag{51}$$

M being a constant (which may depend on d).

By HG, the expectation of $\mathrm{card}(S(\omega, \alpha))$ is 1, hence if $\phi(\omega, \alpha)$ is the radius of the smallest cube centered at 0 and containing $S(\omega, \alpha)$, (50) is satisfied. (49) is satisfied too: Indeed, we need prove that a.a. $(\omega', \alpha') \in \Omega \times \mathbf{Z}_2^d$ is a member of the left-hand side of (49). There is a edge emanating from 0 $(0, -x) \in F(\omega', \alpha')$. Invariance (6) implies $(x, 0) \in F(T^{-x}\omega', \alpha' - x)$, hence $x \in S(T^{-x}\omega', \alpha' - x)$, thus $\|x\| \le n$ where

$$n = \phi(T^{-x}\omega', \alpha' - x) = \max_{z \in S(T^{-x}\omega', \alpha' - x)} \|z\|$$

and $(\omega', \alpha') = T^x(T^{-x}\omega', \alpha' - x) \in T^x\{\phi = n\}$, verifying (49).

We proceed to construct a graph $F(\omega, \alpha)$ with the "thickness" property (51). (The graph should depend on (ω, α) invariantly and measurably.) We have to specify the target of the edge emanating from some $x \in \mathbf{Z}_2^d$. As mentioned above, α may be viewed as a hierarchy of partitions of $\mathbf{Z}_2^d$ into cubes, the cubes of the n-th partition (which have side 2^n) will be called n-cubes.

The set $\mathcal{O}_\omega E = \{x \in G : x\omega \in E\}$ is a.e. non-void, *satur* E being conull.

We may restrict ourselves to α belonging to the conull set $(\mathbf{Z}_2 \setminus \mathbf{Z})^d$. For such α, for any $x \in \mathbf{Z}^d$ the union of cubes containing x for all the partitions in the hierarchy is $G = \mathbf{Z}^d$.

The target of the edges emanating from the points $x \in G$ will be found in steps: In the n-th step, we review the cubes of the n-th partition. For each such cube that intersects $\mathcal{O}_\omega E$, take, say, the first z in the intersection w.r.t. the lexicographic ordering, and let all points in the cube at which the target had not been defined in previous steps be given the target z.

For $\omega \in E$, this makes $S(\omega, \alpha)$, i.e. the set of $x \in G$ with edge going to 0, a subset of some n-cube which contains an $n-1$ -subcube (n being the last step in which 0 was designated as a target – for $\alpha \in (\mathbf{Z}_2 \setminus \mathbf{Z})^d$ there is always such a last step if 0 is not the first element of $\mathcal{O}_\omega E$ lexicographically, which happens a.a. by the argument of Poincaré's recurrence (see §1.1)).

Thus (51) is satisfied with M depending only on d, and we are done.

5.3 Applications to The Renewal Theorem

We shall sketch how formulas obtained from CHG can be applied to prove some classical renewal limit theorems (see [Fel], [Du], [Li], [Ne-P], [DV]).[9] The proofs will be presented for the classical case of stationary independent renewal times with finite expectation, both for nonperiodic continuous Time and for discrete Time (where these two cases will be treated completely analogously). It seems that one can extend the proofs to apply to the stationary case with more general "Time" (i.e. acting groups other than $\mathbf{Z}$ and $\mathbf{R}$) and assume much less than independence, but in these settings one has to make some technical assumptions to make the proof work. Anyhow, we shall make restrictive assumptions only when we need them.

Let G be the acting group, assumed *abelian*, which eventually will be $\mathbf{Z}$ or $\mathbf{R}$. Denote by count the counting measure on G and by dx a Haar measure in G: Lebesgue in the case $G = \mathbf{R}$ and $\mathrm{count}_{\mathbf{Z}}$ in the case $G = \mathbf{Z}$. The mass of a measurable set $S \subset G$ w.r.t. dx will be denoted by $|S|$.

We are given a probability space (E, ν) where $E = \left(\mathbf{R}^+\right)^{\mathbf{Z}}$ with the usual Borel structure, and ν is stationary, i.e. shift-invariant. Denote by X_n the stochastic variable equal to the n-th coordinate, and define $S_n = \sum_{1 \le j \le n} X_j$, $S_{-n} = \sum_{1 \le j \le n} X_{-j}$ for $n \ge 0$. The X_n are the **renewal times** and S_n **is the time of the n'th renewal**. Assume also that each X_n is a.s. positive, and that a.s. $\lim_{n \to \infty} S_n = +\infty$, $\lim_{n \to -\infty} S_n = -\infty$. Identifying $\omega \in E$ with $\{S_n : n \in \mathbf{Z}\}$, E can be identified (after taking away a null subset, if necessary) with the set of all discrete subsets of $G = \mathbf{R}$ or $\mathbf{Z}$ unbounded below and above and containing 0 and as such it is a subset of $\Omega =$ the set of all discrete subsets of G unbounded below and above. On the latter G acts by translations: $x\omega := \omega - x$ and there is a unique $\mathbf{R}$-invariant measure μ on Ω (not necessarily probability or finite, but it will be shown below that μ is σ-finite), s.t. $\nu = \frac{d\mathrm{count}}{dx} \mu|_E$. In other words, ν is the Palm measure for μ (see §2.5). μ can be obtained as in Rmk. 4.3.5 or as in [Ne] (see the proof of f. in Thm. 1.2.8).

Note that since $E = \{\omega \in \Omega : 0 \in \omega\}$, we have that $\forall\, \omega \in \Omega\ \omega$, as a subset of $\mathbf{R}$, is identical with $\mathcal{O}_\omega E$.

The renewal limit theorems that we have in mind state that, under some assumptions, the ν-integral of the sum over $\mathcal{O}_\omega E (= \omega)$ of a translation $x \mapsto h(a^{-1}x)$ of a fixed Borel function $h : E \to \overline{\mathbf{R}^+}$ tends as $a \to +\infty$ to $(\mu(\Omega))^{-1} \int h(x)\, dx$. (Where if $\mu(\Omega) = \infty$ it is understood that $(\mu(\Omega))^{-1} = 0$.) The fact that for $G = \mathbf{R}$ this does not hold for every Borel h is clear if one considers the case when the ranges of all the X_n are countable.

5.3.1 The Case of Mixing G-action

Let us play with CHG to get some formulas (recall that we assumed G abelian):

- Let $h(x)$ be a Borel $\overline{\mathbf{R}^+}$-valued function on G.

  Take the ω-dependent graph, consisting of the 1-simplices $(x, y) \in G^2$ with $y\omega \in E$, weighted by $h(yx^{-1})$. The 1-simplex of measures on G will be taken as (dx, count).

  The source is the function
$$H(\omega) := \sum_{y \in \mathcal{O}_\omega E} h(y).$$

[9] I am indebted to Prof. Jon Aaronson for suggesting to me the possibility of applying methods of this work to renewal problems.

The target is the enhanced function

$$\left(\int h(x)\,dx\right) 1_E(\omega)\cdot\frac{d\text{count}}{dx}.$$

Thus by CHG these two have the same μ-integral, i.e. the μ-integral of H on Ω equals $\int h\,dx$ times $\nu(E)=1$.

- Let $h_1(x)$ and $h_2(x)$ be Borel $\overline{\mathbf{R}^+}$-valued functions on G.

  Take the ω-dependent 2-hypergraph, consisting of the 2-simplices $(x,y,z)\in G^3$ where $y\omega, z\omega\in E$, weighted by $h_1(yx^{-1})h_2(zx^{-1})$. The 2-simplex of measures on G will be $(dx, \text{count}\,, \text{count}\,)$.

  The 0-vertex is the function

  $$\left(\sum_{y\in\mathcal{O}_\omega E} h_1(y)\right)\cdot\left(\sum_{z\in\mathcal{O}_\omega E} h_2(z)\right).$$

  The 1-vertex is the enhanced function

  $$\left(\sum_{z\in\mathcal{O}_\omega E}(h_1 * h_2)(z)\right) 1_E(\omega)\cdot\frac{d\text{count}}{dx},$$

  where $h_1 * h_2$ is the convolution: $(h_1 * h_2)(z) := \int_G h_1(x)h_2(zx)\,dx$.

  Thus by CHG these two have the same μ-integral.

  For $\omega\in\Omega$, let $H_i(\omega) := \sum_{y\in\mathcal{O}_\omega E} h_i(y)$, $i=1,2$.

  Assume $\mu(\Omega)<\infty$. Take h_1 fixed but replace h_2 by the translated $z\mapsto h_2(a^{-1}z)$, $a\in G$. One finds that the ν-integral on $\omega\in E$ of the sum over $\mathcal{O}_\omega E$ $(=\omega)$ of the translated $z\mapsto(h_1 * h_2)(a^{-1}z)$ is equal to the inner product w.r.t. μ on Ω of H_1 and the translated $\omega\mapsto H_2(a\omega)$. If the G-action on Ω is *mixing* this inner product tends, as $a\to\infty$, to $(\mu(\Omega))^{-1}\int H_1\,d\mu\cdot\int H_2\,d\mu$ (assuming, say, that h_1 and h_2 are bounded with compact support). But by the previous item, the latter equals $(\mu(\Omega))^{-1}\int h_1,d\nu\cdot\int h_2,d\nu = (\mu(\Omega))^{-1}\int h_1 * h_2\,d\nu$. Thus we get the well-known fact (see [De]):

  If $\mu(\Omega)<\infty$ then the G-action on Ω being mixing implies holding of the renewal limit theorem for functions h on G which are convolutions of two bounded Borel functions with compact support.

5.3.2 The Nonperiodic Case (for $\mathbf{E}X_1<\infty$)

We use the same notation as in §2.8.4 and §4.3.3, but with "nearest" understood not as in §2.8.4 but as an arbitrary but fixed Borel measurable translation-invariant way to correspond to any closed subset $F\subset G$ and point $x\in G$ a point in F. In fact, our way to do so (for $G=\mathbf{R}$ or $\mathbf{Z}$) will always be to take the point $\max(F\cap\,]-\infty,x])$. Thus, according to the notation there, for $\omega\in\Omega$ $\pi(\omega)=\pi_E(\omega)$ is the last point in $\mathcal{O}_\omega E$ $(=\omega)$ not bigger than 0 and for $\omega\in E$ $P(\omega)=P_E(\omega)$ is the set of $x\in G$ with 0 as the last point in $\mathcal{O}_\omega E$ not bigger than x, that is, $P(\omega)=[0,\min(\,]0,\infty]\cap\mathcal{O}_\omega E)[\;=[0,X_1(\omega)[$. But for the time being, assume "nearest" to be interpreted in an arbitrary way as above.

Since we assume G is abelian, we use additive notation for G, compelling us to write $T^x\omega$ instead of $x\omega$.

Let us "play" with CHG:

Take the ω-dependent graph (cf. (31))

$$\begin{aligned}&\{(x,y)\in G^2 : y \text{ is the "nearest" point to } x \text{ in } \mathcal{O}_\omega E(=\omega)\} =\\ &=\{(x,y)\in G^2 : y-x=\pi_E(T^x\omega)\}=\{(x,y)\in G^2 : T^y\omega\in E,\ x-y\in P_E(T^y\omega)\},\end{aligned}$$

and the 1-simplex of measures on G $\quad(dx, \text{count}\,)$.

The source is the function 1. The target is the enhanced function $|P_E(\omega)|1_E(\omega)\frac{d\text{count}}{dx}$. Hence by CHG:

$$\mu(\Omega) = \int_E |P_E(\omega)| \frac{d\text{count}}{dx}\, d\mu(\omega) = \int_E |P_E(\omega)|\, d\nu(\omega) \tag{52}$$

Note that both sides may be $+\infty$, yet if we take in the above graph only the edges (x, y) with $y - x \in K_n$ where K_n are compact with union G, we deduce, using CHG, that μ on Ω is *σ-finite*: Ω is the union of the sets with finite μ-measure $\{\omega : \pi(\omega) \in K_n\}$.

We shall assume from now on that $\mu(\Omega) = \int_E |P(\omega)|\, d\nu(\omega)$ is *finite*. For our meaning of "nearest" $P(\omega) = [0, x_1(\omega)[$ this and (52) mean that $\int_E X_1(\omega)\, d\nu(\omega) = \mu(\Omega) < \infty$

Now for $a \in G$, Take the ω-dependent graph

$$\begin{aligned} &\{(x, y) \in G^2 : x\omega \in E, y \text{ is the "nearest" point to } x - a \text{ in } \mathcal{O}_\omega E (= \omega)\} = \\ &= \left\{(x, y) \in G^2 : T^x\omega \in E, y - x = -a + \pi_E\left(T^{x-a}\omega\right)\right\} = \\ &= \left\{(x, y) \in G^2 : T^x\omega, T^y\omega \in E,\ x - y \in a + P_E(T^y\omega)\right\}, \end{aligned}$$

and the 1-simplex of measures (count , count).

The source is the enhanced function $1_E\frac{d\text{count}}{dx}$. The target is the enhanced function

$$\#\omega \cap (a + P_E(\omega))\, \frac{d\text{count}}{dx}.$$

Thus CHG tells us that

$$\int_E \#\omega \cap (a + P_E(\omega))\ d\nu = 1 \tag{53}$$

i.e. that substituting in the renewal limit theorem for $h(x)$ the ω-dependent $1_{p(\omega)}(x)$ gives an identity, rather than a limit: summing the translated function on $\omega = \mathcal{O}_\omega E$ and taking the ν-expectation gives the correct value (at least when $\mu(\Omega) < \infty$):

$$1 = (\mu(\Omega))^{-1} \int_E \left(\int 1_{P(\omega)}(x)\, dx \right) d\nu(\omega)$$

(see (52)).

From this we shall deduce the limit theorem in a "Tauberian" manner.

Denote by $\mathcal{C}_{00} = \mathcal{C}_{00}(G)$ the space of continuous functions on G with compact support.

Our goal is to show that:

$$\lim_{a \to +\infty} \int_E \left(\delta_{-a} * \sum_{x \in \omega} \delta_x \right) d\nu(\omega) = (\mu(\Omega))^{-1}\, dx, \tag{54}$$

the limit taken in the weak topology w.r.t. $\mathcal{C}_{00}$. This will imply holding of the renewal limit assertion for $h(x)$ bounded Riemann-integrable with compact support.

Note, that the measure on G $\sum_{x\in\omega} \delta_x$ gives, for each subset of Time G, the # of renewals in this subset.

We shall make assumptions needed to carry out our proof and show that they hold for the case considered by us.

Assumption 1 *The Borel functions $\omega \mapsto \int_G h(x)\, d\left(\delta_{-a} * \sum_{x\in\omega} \delta_x\right)(x) = \sum_{x\in\omega} h(x - a)$, $a \in G$ form a weakly compact family in $L^1(E, \nu)$ for each fixed $h \in \mathcal{C}_{00}$.*

This assumption holds when $G = \mathbf{R}$ or $\mathbf{Z}$ and the X_n are independent. Indeed, one may consider $h = 1_{[0,c]}$ instead of $h \in \mathcal{C}_{00}$. Now, if S_m is the first one not less than a, then the # of renewals in $[a, a+c]$ is $\leq$ than the # of renewals in $[S_m, S_m + c]$, which by independence has the same distribution function as the # of renewals

in $[0,c]$. The latter does not depend on a, and is an L^1-function of $\omega \in E$, since the probability that the # of renewals in $[0,c] \geq n$ is

$$\begin{aligned} &\Pr(S_n \leq c) \ \leq \Pr\left(\textstyle\sum_{1\leq j\leq n} \min(X_j,1) \leq c\right) = \\ &= \Pr\left(\textstyle\sum_{1\leq j\leq n} \min(X_j,1) - n\mathbf{E}\left(\min(X_1,1)\right) \leq c - n\mathbf{E}\left(\min(X_1,1)\right)\right) = \\ &= O(n^{-2}). \end{aligned}$$

By Assumption 1 the set of measures $\int_E \left(\delta_{-a} * \sum_{x\in\omega}\delta_x\right)\, d\nu(\omega)$, $a \in G$ is bounded on every compact $\subset G$, hence is contained in a compact metrizable set in the weak topology w.r.t. $\mathcal{C}_{00}$. Thus assume that $a_j \to +\infty$ such that $\int_E \left(\delta_{-a_j} * \sum_{x\in\omega}\delta_x\right)\, d\nu(\omega)$ converges in the weak topology w.r.t. $\mathcal{C}_{00}$ to some measure $\tilde\lambda$ on G (necessarily a Radon measure that is uniformly bounded on the translations of any fixed compact set). It suffices to prove that for each such (a_j) $\quad \tilde\lambda = (\mu(\Omega))^{-1}\, dx$.

Assumption 2 *The Hewitt-Savage 0-1 law holds, namely, every permutable event in E, i.e. every event not changed by any permutation of the X_n's that moves only a finite number of indices, has probability* 0 *or* 1.

This holds when the X_n's are independent – see [Du] §3.1.

Now, Assumption 2 implies that for every $h \in \mathcal{C}_{00}$, every limit of a subsequence of

$$\omega \mapsto \int_G h(y)\, d\left(\delta_{-a_j} * \sum_{x\in\omega}\delta_x\right)(y) = \sum_{x\in\omega} h(x-a_j)$$

in the weak topology of $L^1(E,\nu)$, being measurable w.r.t. the σ-algebra of permutable events, must be a.e. constant, necessarily equal to the limit of the ν-integrals $\left(\int_G h(x)\, d\tilde\lambda(x)\right)\cdot 1$. By Assumption 1, the latter is the weak L^1-limit of $\omega \mapsto \sum_{x\in\omega} h(x-a_j)$. This means that for every function F on $E\times G$ which is of the form $(\omega,x)\mapsto f(\omega)h(x)$, $f \in L^\infty(E)$, $h\in\mathcal{C}_{00}$, one has

$$\lim_j \int_E \sum_{x\in\omega} F(\omega, x-a_j)\, d\nu(\omega) = \int_G\int_E F(\omega,x)\, d\nu(\omega)\, d\tilde\lambda(x). \tag{55}$$

We would like to have (55) for $F(\omega,x) = 1_{a+P(\omega)}(x)$, so that we may compare (53) and (55).

In the set of open (resp. closed) subsets of Ω we take the Effros Borel structure (see §2.7), given by identifying each open set $S\subset G$ (resp. each closed $S\subset G$) with the set $\{u\in\mathcal{U} : u\subset S\}\subset 2^{\mathcal{U}}$ (resp. $\{u\in\mathcal{U} : u\cap S=\emptyset\}\subset 2^{\mathcal{U}}$), where $\mathcal{U}$ is some countable base to the topology.

Assumption 3 *For ν-a.a. $\omega\in E$ $|\partial(P(\omega))| = 0$ and the mappings sending $\omega\in E$ to the interior $P(\omega)^\circ$, resp. to the closure $\overline{P(\omega)}$, are measurable.*

This assumption clearly holds for our case $P(\omega) = [0, X_1(\omega)[$.

From Assumption 3 one deduces that the function on $E\times G$ $\quad(\omega,x)\mapsto 1_{P(\omega)^\circ}(x)$ is a supremum of countably many finite linear combinations of functions of the form $(\omega,x)\mapsto 1_S(\omega)h(x)$, $S\subset E$ measurable and $h\in\mathcal{C}_{00}$, hence is a limit of an increasing sequence (F_k) of such combinations. (Note that the collection of such linear combinations is stable w.r.t. the lattice operations min and max.) By (55) we have, for any $a\in G$:

$$\lim_j \int_E \sum_{x\in\omega} F_k(\omega, x-a-a_j)\, d\nu(\omega) = \int_G\int_E F_k(\omega,x-a)\, d\nu(\omega)\, d\tilde\lambda(x).$$

Going to the limit in k one obtains:

$$\lim_j \int_E \#\left(\omega\cap(a+a_j+(P(\omega))^\circ\right)\, d\nu(\omega) \geq \int_G\int_E 1_{P(\omega)^\circ}(x-a)\, d\nu(\omega)\, d\tilde\lambda(x)$$

and taking into account (53), one has

$$1 \geq \int_G \nu\{\omega : x-a\in P(\omega)^\circ\}\, d\tilde\lambda(x).$$

The last expression, as a function of a, is the convolution of $\tilde{\lambda}$ with the function $x \mapsto \nu\{\omega : x \in P(\omega)^\circ\}$. By Assumption 3, this function is equal dx-a.e. to $x \mapsto \nu\{\omega : x \in P(\omega)\}$, therefore its convolution with $\tilde{\lambda}$ is equal dx-a.e. to $\tilde{\lambda} * (x \mapsto \nu\{\omega : x \in P(\omega)\})$. Thus we conclude:

$$1 \geq \tilde{\lambda} * (x \mapsto \nu\{\omega : x \in P(\omega)\}) \qquad dx - \text{a.e.} \tag{56}$$

If we knew that $\forall \omega P(\omega) \subset$ a fixed compact K, we could similarly prove the opposite inequality by considering $U \setminus P(\omega)$ instead of $P(\omega)$ where U is fixed open, relatively compact and contains K. But that need not be the case. Clearly, we shall still have the opposite inequality if the following assumption holds:

Assumption 4 *For every $\varepsilon > 0$ $\exists$ is a relatively compact set K s.t. for all $a \in G$ close enough to $+\infty$*

$$\int_E \#\omega \cap (a + (P(\omega) \setminus K)) \, d\nu \leq \varepsilon \tag{57}$$

Using CHG, one can transform the left-hand side of (52): take the ω-dependent graph

$$\begin{gathered}\left\{(x,y) \in G^2 : T^x\omega, T^y\omega \in E,\, x - y - a \in P(T^y\omega) \setminus K\right\} = \\ = \left\{(x,y) \in G^2 : T^x\omega \in E, y - x + a = \pi_E\left(T^{x-a}\omega\right) \notin -K\right\}\end{gathered}$$

and the 1-simplex of measures (count , count).

The source is the enhanced function

$$1_{\{\omega \in E : \pi(T^{-a}\omega) \notin -K\}} \frac{d\text{count}}{dx};$$

the target is the enhanced function

$$\#\omega \cap (a + (P(\omega) \setminus K)) \frac{d\text{count}}{dx};$$

and by CHG we can write (57) in the equivalent form

$$\nu\{\omega \in E : \pi(T^{-a}\omega) \notin -K\} < \varepsilon \tag{58}$$

Let us show that if $G = \mathbf{R}$ or $\mathbf{Z}$ and the X_n are independent, and $\mu(\Omega) = \int_E X_1(\omega) \, d\nu(\omega) < \infty$, and one takes $P(\omega) = [0, X_1(\omega)[$ then Assumption 4 holds, where we use the form (58). Indeed, what we have to prove amounts to showing that the ν-probability that the last renewal before time $-a$ was even before $-a-b$ (i.e. that $\omega \cap [-a-b, -a[= \emptyset$), tends to 0 when $b \to +\infty$ uniformly in a for a near $+\infty$ (we take $a > 0$). But by independence of the X_n,

$$\begin{aligned}&\Pr(\omega \cap [-a-b, -a[= \emptyset) \leq \textstyle\sum_{k \in \mathbf{Z}^+} \sum_{n \in \mathbf{Z}^+} \Pr\left(S_{-n}(\omega) \in [-a+k, -a+k+1[\wedge X_{-(n+1)} > b+k\right) \leq \\ &\leq \textstyle\sum_{k \in \mathbf{Z}^+} \left(\mathbf{E}\#\omega \cap [-a+k, -a+k+1[\right) \Pr(X_1 > b+k) \leq \text{const} \cdot \sum_{k \in \mathbf{Z}^+} \Pr(X_1 > b+k) \to_{b \to +\infty} 0\end{aligned}$$

where in the last inequality we used Assumption 1 to get the bound const and also the fact that X_1 is integrable.

Thus we finally have:

$$\tilde{\lambda} * (x \mapsto \nu\{\omega : x \in P(\omega)\}) = 1 \qquad dx - \text{a.e.} \tag{59}$$

Note that the function on G $\quad x \mapsto \nu\{\omega : x \in P(\omega)\}$ has dx-integral equal to $\int_E |P(\omega)| \, d\nu(\omega) = \mu(\Omega) < \infty$. Suppose the following assumption holds:

Assumption 5 *The fourier transform of the nonnegative L^1 function on G $\quad x \mapsto \nu\{\omega : x \in P(\omega)$ never vanishes.*

Then this assumption together with (59) would imply that $\tilde{\lambda}$ is dx multiplied by the reciprocal of the value at 0 of the Fourier transform of $f(x) = \nu\{\omega : x \in P(\omega)\}$, namely by the reciprocal of $\int_G \nu\{\omega : x \in P(\omega)\}\, dx = \mu(\Omega)$, and we would be done. This implication obtains as follows:

We wish to infer from the fact that $\tilde{\lambda}$ and $(\mu(\Omega))^{-1}\, dx$ have the same convolution with our $f \in L^1(G, dx)$, whose Fourier transform $\hat{f}(t)$ never vanishes, that $\tilde{\lambda} = (\mu(\Omega))^{-1}\, dx$. Now, $\tilde{\lambda}$ can be weakly approximated by its convolutions with functions in $\mathcal{C}_{00}$, which are bounded continuous functions on G (Recall that $\tilde{\lambda}$ is uniformly bounded on the translates of any fixed compact $\subset G$). Thus, it suffices to prove that if $\ell(x)$ is continuous bounded on G and $\ell * f = 1$ then $\ell \equiv (\mu(\Omega))^{-1} \cdot 1$. This follows from the fact that the closed ideal in $L^1(G)$ (where the multiplication is convolution and we take the norm topology) generated by f is the whole $L^1(G)$ (this is Wiener's Tauberian Theorem). This follows from Fourier transform considerations: Indeed, the said ideal contains the functions $k(x) \in L^1(G)$ whose fourier transforms are of the form $\hat{k_1}(t)\hat{f}(t)$ where $k_1 \in L^1(G)$ with $\hat{k_1} \in \mathcal{C}_{00}(\hat{G})$. Since $\hat{f}$ never vanishes, every function in $L^1(G)$ with Fourier transform in $\mathcal{C}_{00}(\hat{G})$ can be approximated by such k, hence belongs to the ideal and the latter functions are dense in $L^1(G)$.

Thus it remains to ensure that Assumption 5 holds when we take $G = \mathbf{R}$ or $\mathbf{Z}$ and $P(\omega) = [0, X_1(\omega)[$. But then $\nu\{x \in P(\omega)\} = \Pr\{X_1(\omega) > x \geq 0\}$ which is 0 for $x < 0$ and is nonincreasing nonnegative for $x \geq 0$. If t is a nonzero element of the dual group $\hat{G}$, we have, if $G = \mathbf{R}$, $\hat{G} = \mathbf{R}$ and dx is the Lebesgue measure:

$$\int_0^\infty \exp(2\pi ixt) \Pr\{X_1 > x\}\, dx = (2\pi it)^{-1} \int_0^\infty (1 - \exp(2\pi ixt))\ d\Pr\{X_1 > x\},$$

and if $G = \mathbf{Z}$, $\hat{G} = \mathbf{R}/\mathbf{Z}$ and dx is the counting measure:

$$\sum_{0 \leq x < \infty} \exp(2\pi ixt) \Pr\{X_1 > x\} = (1 - exp(2\pi t))^{-1} \sum_{0 \leq x < \infty} (1 - exp(2\pi ixt)) \left(\Pr\{X_1 > x - 1\} - \Pr\{X_1 > x\}\right).$$

Therefore the Fourier transform of $x \mapsto \Pr\{X_1 > x\}$ can vanish at t only if the distribution of X_1 is concentrated in $t^{-1}\mathbf{Z}$. Thus, in the nonperiodic case, i.e. when there is no proper closed subgroup $H \subset G$ s.t. a.s. $X_1 \in H$, Assumption 5 holds and we have the renewal limit assertion.

A Appendices

A.1 Generation of Measures via Given "Preintegrable" Functions

We describe a way to obtain a measure on a set Ω, which we use in §2.4. This method seems convenient when a measure has to be constructed by some "integration" of a family of given measures.

The starting point is a set Pre of $[0,\infty]$-valued functions on Ω, called **preintegrable**, with a functional (**integral**) $\mathcal{I} : \mathrm{Pre} \to [0,\infty[$, satisfying the following assumptions:

1. Pre is a cone, i.e. Pre contains 0 and is stable w.r.t. addition and multiplication by finite nonnegative real constants, and $\mathcal{I}$ is additive and non-negatively linear.

2. If $f, g : \Omega \to \overline{\mathbf{R}^+}$ s.t. $f, f+g \in \mathrm{Pre}$ then $g \in \mathrm{Pre}$. (Consequently, $f, g \in \mathrm{Pre}$, $f \geq g \Rightarrow \mathcal{I}(f) \geq \mathcal{I}(g)$.)

3. If $f_n \in \mathrm{Pre}$, $n \in \mathbf{N}$, $f_n \uparrow$ and $\mathcal{I}(f_n)$ is bounded, then $\lim f_n \in \mathrm{Pre}$ and $\mathcal{I}(\lim f_n) = \lim(\mathcal{I}(f_n))$.

4. If $f_0 \in \mathrm{Pre}$ and $\mathcal{I}(f_0) = 0$ then any function $f \leq f_0$ belongs to Pre.

Now say that a set $E \subset \Omega$ is **measurable** if

$$\forall f \in \mathrm{Pre}\;\; f \cdot 1_E \in \mathrm{Pre}.$$

One proves easily, using the above assumptions, that the measurable sets form a σ-algebra. **Measurable functions** will be functions measurable w.r.t. this σ-algebra. Note that if g is measurable and bounded, then

$$\forall f \in \mathrm{Pre}\;\; f \cdot g \in \mathrm{Pre}$$

.

To define the measure μ on this σ-algebra, a measurable set E will have finite measure iff $1_E \in \mathrm{Pre}$ and then $\mu(E) := \mathcal{I}(1_E)$. Otherwise $\mu(E) = \infty$. The assumptions on Pre and $\mathcal{I}$ imply readily that μ is σ-additive. (Moreover, by 4. μ is complete, i.e. every subset of a set of measure 0 is measurable.) Thus $\int d\mu$ is defined. As usual, a function $f \geq 0$ is **integrable** if it is measurable and has finite integral.

An important fact is that any integrable function $f \geq 0$ is preintegrable, and any measurable preintegrable function $f \geq 0$ is integrable, and then $\mathcal{I}(f)$ and $\int f\, d\mu$ coincide. (Thus, to find the integral of a measurable function $f \geq 0$ one just checks if f is in Pre. If it is, its integral is $\mathcal{I}(f)$, otherwise $\int f\, d\mu = \infty$).

Indeed, Note first that any $\{0,\infty\}$-valued $f \in \mathrm{Pre}$ must have $\mathcal{I}(f) = 0$, since $\mathcal{I}(f)$ is finite and $2\mathcal{I}(f) = \mathcal{I}(2f) = \mathcal{I}(f)$. Therefore if $f \geq 0$ is measurable preintegrable, then $\{f = +\infty\}$ is null. Now an integrable $f \geq 0$ can be obtained from characteristic functions of sets of finite measure by addition and increasing limits with bounded integral, hence it is in Pre. On the other hand, if $f \geq 0$ is measurable preintegrable, and $a > 0$, then $E = \{\infty > f > a\}$ is measurable. 1_E is of the form $f \cdot g$ where g is measurable bounded, hence 1_E is preintegrable, implying $\mu(E) = \mathcal{I}(1_E) < \infty$. From that one easily deduces f integrable and $\int f\, d\mu = \mathcal{I}(f)$.

A.2 Standard Borel Spaces and Products of Two Standard Spaces

Recall that a **standard Borel space** is a Borel space which is isomorphic, as a Borel space, to a Lusin topological space (recall that in any topological space the Borel structure understood is the σ-algebra of "ordinary" Borel subsets). We shall use facts about Lusin and Polish spaces – see [Bo-T] Ch. IX §6, [Ke-D], [Ku] (where the terminology is slightly different). Recall, in particular, that topological spaces where the topology can be given by a complete saparable metric are called **Polish spaces**; Topological spaces which are continuous 1-1 images of Polish spaces are called **Lusin spaces**; a subset of a Lusin space is Borel iff it is Lusin in its relative topology; a subset of a Polish space is G_δ iff it is Polish in its relative topology; for any 1-1 Borel mapping between standard Borel spaces the image is Borel and the mapping is an isomorphism of the Borel structures (with the image); any Lusin space is a 1-1 continuous image of a 0-dimensional polish space, i.e. a Polish space with a base to the topology consisting of clopens; any Polish space can be continuously embedded in a metric compact space, the latter can be chosen 0-dimensional if the former is.

In fact, two standard spaces of the same cardinality are isomorphic as Borel spaces. Thus the only isomorphism types of standard Borel spaces are: finite sets, the type of a countable set with the σ-algebra of all subsets, and the unique type of a standard space of the cardinality of the continuum. Thus for many purposes one may assume the latter is the unit interval with Borel subsets. However, for the purposes below it is preferable to consider the totality of all Lusin topologies in the standard space, having in mind, of course, topologies s.t. their Borel structure is the given one.

In this vein, one notes that for every countable Boolean algebra of subsets (of a standard Borel space) which separates points, the obvious mapping to $2^{\mathbf{N}}$ is 1-1 Borel, hence a Borel isomorphism with the image, which is Borel in $2^{\mathbf{N}}$ hence Lusin in the relative topology. This implies that for every countable set of Borel sets there is a Polish topology where all are clopen, consequently for every countable collection of Borel bounded real-valued functions one may find a Polish topology where all are continuous, and for every countable collection of $[0,\infty]$-valued Borel functions there is a Polish topology where all are l.s.c. (lower semi-continuous).

Note that by using the diagonal in a countable product, one proves that for every countable family of Lusin (resp. Polish, resp. 0-dimensional Polish) topologies there is a topology of the same kind finer than all of them.

Thus when one is confronted with, say, a non-negative Borel function on a standard space, one may assume that it is l.s.c. for some Polish topology there.

Matters are not so simple if one deals with *a product of two standard spaces X and Y* and one may choose topologies in X and Y, but in the product one always take the *product topology.*

Proposition A.2.1 *Let X and Y be standard Borel spaces. Let $E \subset X \times Y$ be Borel. T.f.a.e:*

(i) E is a disjoint union of countably many "Borel rectangles": products of Borel sets in X and Y

(ii) E is open in some product of Lusin topologies in X and Y.

Proof **(i)** $\Rightarrow$ **(ii)**: take in X and Y topologies making all sides of the rectangles open.

(ii) $\Rightarrow$ **(i)**: Since every Lusin space is a continuous 1-1 image of a Polish space, one may assume the topologies are Polish. Choose countable bases to the topologies and consider the countable Boolean algebras generated by the bases. E is a countable union of "rectangles" with sides belonging to the Boolean algebras, hence a countable disjoint union of such.

QED

As an example of a set which does not satisfy (i) and (ii) in the previous proposition, take the diagonal in $[0,1] \times [0,1]$.

Proposition A.2.2 *Let X and Y be standard Borel spaces. Let $f : X \times Y \to [0,\infty]$ be Borel. T.f.a.e:*

(i) f can be represented as a series:

$$f(x,y) = \sum_{i \geq 1} g_i(x) h_j(y) \quad x \in X,\ y \in Y$$

where $g_i : X \to [0,\infty]$, $h_i : Y \to [0,\infty]$ are Borel.

(ii) f is l.s.c. (lower semi-continuous) for the product of some Lusin topologies in X and Y.

(iii) Every set $\{f > a\}$, $a \in [0,\infty[$ satisfies the requirements of the previous proposition.

Proof **(ii)** $\Rightarrow$ **(iii)** is obvious.

(i) $\Rightarrow$ **(ii)**: there are Lusin topologies in X and Y making all g_i and h_i l.s.c., thus making f l.s.c.

(iii) $\Rightarrow$ **(i)**: take all sets $\{f > a\}$ for *dyadic* a, describe them as disjoint unions of "Borel rectangles" and consider the countable Boolean algebras in X and Y generated by all their sides. f is a supremum of countable positive combinations of characteristic functions of countable unions of rectangles with sides in the

Boolean algebras, hence a supremum of countable positive combinations of characteristic functions of single rectangles, and since the latter combinations are stable w.r.t. lattice operations and subtraction, f is a sum of a series of positive multiples of characteristic functions of rectangles.

QED

Again, the characteristic function of the diagonal in $[0,1]\times[0,1]$ does not satisfy the requirements of Prop. A.2.2.

A.3 Converting Measurable Action to Continuous Action

Our setting is a 2nd-countable locally compact group G acting in a Borel manner on a standard Borel space $(\Omega,\mathcal{B})$, thus making it into a *standard G-space*.

A special case of the above is a **metrizable compact G-space**, where one takes usual Borel sets and $(x,\omega)\mapsto x\omega$ is assumed *continuous in the two variables*.

There is a well-known method ([Va], see also, e.g. [AK], [Do], [Ma] where the idea of mapping ω to the function on G $\mathcal{O}_\omega f$ is employed) to embed any standard G-space as a Borel subset of a metrizable compact G-space (this is done in a definitely *non-unique* way).

Choose any countable set $\mathcal{F}$ of Borel functions f with $|f|\leq 1$, separating points in Ω (this exists by standardness). For any $\omega\in\Omega$ and any $f\in\mathcal{F}$ we have

$$\mathcal{O}_\omega f := x\mapsto f(x\omega),\ \text{thus}\ \mathcal{O}_\omega f: G\to\mathbf{R}$$

This may be considered as an element of the unit ball of $\mathcal{B}(L^\infty(G))$, the latter taken w.r.t. (right or left) Haar measure, and is endowed with the w^*-topology from L^1. This unit ball is metrizable compact and the mapping

$$\omega\mapsto(\mathcal{O}_\omega f)_{f\in\mathcal{F}}$$

maps Ω into the compact metrizable $K=\mathcal{B}(L^\infty(G))^{\mathcal{F}}$, mapping the G-action into right translation in any coordinate $\mathcal{B}(L^\infty(G))$.

Now we can verify some facts:

The action of G by right translation in any $\mathcal{B}(L^\infty(G))$ (hence in a power $\mathcal{B}(L^\infty(G))^{\mathcal{F}}$) is continuous in the two variables.

Our map from Ω into the power is Borel. Indeed, the Borel structure in $\mathcal{B}(L^\infty(G))$ is defined by some countable set of "coordinates" $h\in L^1(G)$, and for such h, the function (λ is some Haar measure)

$$\omega\mapsto\langle\mathcal{O}_\omega f,h\rangle=\int f(x\omega)h(x)\,d\lambda$$

is Borel.

This map is 1-1. Indeed, suppose ω_1 and ω_2 map to the same

$$(\mathcal{O}_{\omega_1}f)_{f\in\mathcal{F}}=(\mathcal{O}_{\omega_2}f)_{f\in\mathcal{F}}\,.$$

(where equality of the $\mathcal{O}_\omega f$ means equality in L^∞, that is equality a.e. w.r.t. Haar). This means that for a.a. $x\in G$ $\forall f\in\mathcal{F}$ $(\mathcal{O}_{\omega_1}f)(x)=(\mathcal{O}_{\omega_2}f)(x)$, i.e. $\forall f\in\mathcal{F}$ $f(x\omega_1)=f(x\omega_2)$. Hence this holds for *some* x, which means, since $\mathcal{F}$ separates points, that $x\omega_1=x\omega_2$, implying $\omega_1=x^{-1}x\omega_1=x^{-1}x\omega_2=\omega_2$.

Since the map is 1-1 Borel between standard spaces, its image is a Borel subset of the metrizable compact K (see §A.2).

Instead of the above K, we may and do take as our K the closure of the image of Ω.

Note that the relative topology from K is a Lusin topology in Ω itself (see §A.2) making the G-action continuous in the two variables.

Note that if $f\in\mathcal{F}$ and one takes a convolution of f with an L^1-functionon G (i.e. one considers $\omega\mapsto\int_G f(x\omega)\,d\nu(x)$, when ν is a finite measure absolutely continuous w.r.t. Haar), then the latter (extends to) a continuous function on K. If G is discrete one may take $\nu=\delta_e$, and any $f\in\mathcal{F}$ extends to a continuous function on K. Thus *for discrete G* K may be tuned so that any countably many given bounded Borel

functions extend to continuous functions on K, hence so that any countably many given $[0,\infty]$-valued Borel functions extend to l.s.c. (lower semi-continuous) on K (here the extension need not be unique).

If Ω was from the start a dense invariant Borel subset of a metrizable compact G-space K_0, we can take all the $f \in \mathcal{F}$ continuous on K_0, and then we have a mapping from K_0 into the power $K = \mathcal{B}(L^\infty(G))^{\mathcal{F}}$ which is 1-1 continuous, hence an isomorphism with a compact sub-G-space. This shows that by this construction one gets all embeddings of Ω as a dense subset of a metrizable compact G-space up to isomorphism.

These metrizable compact G-spaces in which we embedded a standard Borel G-space are not canonical, since a countable set of f's has to be chosen. One may get a *canonical, but not metrizable* G-compact by taking the set $\mathcal{F}$ of *all* Borel functions f with $|f| \leq 1$, and considering the closure $\mathcal{K}(\Omega)$ of the image of Ω in the power $\mathcal{B}(L^\infty(G))^{\mathcal{F}}$. The Borel sets in Ω will be the intersection of Ω with the *Baire* sets in $\mathcal{K}$.

For G countable discrete, $\mathcal{K}$ does not depend on G and is just the Stone space of the σ-algebra of Borel sets in Ω. In particular, in this case every bounded Borel function on Ω extends to a continuous function on $\mathcal{K}$, and every unbounded non-negative Borel function extends to a b.l.s.c. function on $\mathcal{K}$ (see §3.2).

For general G this is not the case: a necessary condition for a Borel f on Ω to extend to a continuous function on $\mathcal{K}$ is that $\mathcal{O}_\omega f$ is continuous on G for all $\omega \in \Omega$. On the other hand, any convolution of a Borel function on Ω with some L^1 function on G does extend to a continuous function on $\mathcal{K}$.

A.4 Mean Ergodic Theorems for General (Discrete) Groups

The aim of this § is to give a treatment, in the spirit of §3.5, of the well-known derivation of mean ergodic theorems for general groups, using weak compactness and Ryll-Nardzewski's fixed point Theorem. (see [BA], [GD-A], [GD-D], [Gr], [J] §2, [Tr-E], [Tr-S]).

G is a (discrete) group, in general *non-amenable*.

Consider a G**-normed space** V, i.e. V is a G-vector space (that is: G acts on V linearly), normed by an invariant norm $\|\|$. Recall the definition of averages in §3.5

Definition A.4.1 *(cf. [BA]) We say that a $v \in V$ converges to a $v_0 \in V$ in the* **Accumulating Averages (AA)** *sense or that v_0 is the AA-limit of v (denote: $v \overset{AA}{\to} v_0$ or $v_0 = \lim_{AA} v$), if v_0* is G-invariant *and*

for every average v' of v, every neighbourhood of v_0 contains an average of v'.

It is straightforward that the AA-limit of v is *unique*.

Also, $v \overset{\text{AA}}{\to} v_0 \Leftrightarrow v - v_0 \overset{\text{AA}}{\to} 0$.

We have: if $\lim_{\text{AA}} v$ exists, then for any G-invariant $v^* \in V^*$, $\langle v^*, v\rangle = \left\langle v^*, \lim_{\text{AA}} v\right\rangle$.

We use the following notation: $\mathcal{AV}(v)$ is the set of averages of a vector v,

$$\|v\|_A = \inf_{w\in\mathcal{AV}(v)} \|w\|$$
$$\|v\|_{AA} = \sup_{w\in\mathcal{AV}(v)} \|w\|_A$$

Thus, $v \overset{\text{AA}}{\to} v_0 \Leftrightarrow \|v - v_0\|_{AA} = 0$.

Clearly, $\|v\|_A \leq \|v\|_{AA}$.

If $\lim_{\text{AA}} v$ exists then $\|\lim_{\text{AA}} v\| \leq \|v\|_A$

Proposition A.4.2 *a.* $\|v + v'\|_A \leq \|v\|_{AA} + \|v'\|_A$

b. $\|v + v'\|_{AA} \leq \|v\|_{AA} + \|v'\|_{AA}$

Proof Call an operator of the form $v \mapsto \sum_x \lambda_x xv$ (finite sum) where $\forall x\ \lambda_x \geq 0\ \sum_x \lambda_x = 1$ an *averaging operator (a.o.)*. These operators form a semigroup. In the rest of the proof, L, L' and L'' will refer to a.o.'s. We have $\forall L, \|Lv\| \leq \|v\|$ and $\forall L, \|L(v + v')\| \leq \|L(v)\| + \|L(v')\|$

Proof of a.:

Choose L with $\|Lv\| \leq \|v\|_A + \varepsilon$. Choose L' with $\|L'Lv'\| \leq \|v'\|_{AA} + \varepsilon$. Then $\|L'L(v + v')\| \leq \|v\|_A + \|v'\|_{AA} + 2\varepsilon$.

Proof of b.:
$\forall\ L''$, Choose L with $\|LL''v\| \leq \|v\|_{AA} + \varepsilon$. Choose L' with $\|L'LL''v'\| \leq \|v'\|_{AA} + \varepsilon$. Then $\|L'LL''(v + v')\| \leq \|v\|_{AA} + \|v'\|_{AA} + 2\varepsilon$.
QED

Thus, $\|\|_{AA}$ is a semi-norm dominated by $\|\|$.

This implies that AA-convergence has the desired properties: if $v \xrightarrow{\text{AA}} v_0$ and $w \xrightarrow{\text{AA}} w_0$ then $\alpha v + \beta w \xrightarrow{\text{AA}} \alpha v_0 + \beta w_0$.

That much cannot be said for the property: every neighbourhood of v_0 contains an average of v (i.e. $\|v - v_0\|_A = 0$). Note that while for abelian G the averaging operators commute, hence any two averages have a common average, which implies immediately $\|v\|_{AA} = \|v\|_A$, for non-abelian groups two averages need not have a common average.

Example A.4.3 $G =$ the infinite dihedral group D_∞ (which is *amenable*), realized as the set of the transformations of $\mathbf{R}$ generated by $x_1 : t \to t+1$ and $x_- : t \to -t$. $V =$ the space of polynomials of degree ≤ 3. $v = t(t-1)(t+1)$. $x_- v = -v$ so $w = \frac{1}{2}(v(t+1) - v) = \frac{3}{2}t(t-1)$ and $-w = \frac{1}{2}(-v(t+1) + v)$ are averages of v which have no common average (any average of w has coefficient $\frac{3}{2}$ at t^2).

Example A.4.4 Let G be a non-amenable group. Let K be the set of averages on $\ell^\infty(G)$, i.e. the set of positive functionals on $\ell^\infty(G)$ giving the value 1 to the constant sequence 1. In Ω, take the w^*-topology from ℓ^∞. K is a G-convex compact space. Since G is non-amenable, there is no G-fixed point in K, hence there is a minimal non-empty convex subset $K_0 \subset K$. K_0 is not a singleton. It satisfies: for every $\omega \in K_0$, the closed convex hull of its orbit is the whole K_0 (if G is countable, K_0 has a compact *metrizable* factor with the same property).

Let $f : K_0 \to \mathbf{R}$ be non-constant continuous affine with minimum 0 and maximum $a > 0$. By Thm. 3.5.4 b. (for $\mathcal{C}(K_0)$ and $p = \max$), f has averages with maximum arbitrarily close to 0, and averages with minimum arbitrarily close to a (a property inherited by all their averages). Thus $\|f\|_{\mathcal{C}(K_0),A} = 0$ while $\|f\|_{\mathcal{C}(K_0),AA} = a$.

Note that by Thm. A.4.7, for *amenable* G we have always $\|v\|_A = \|v\|_{AA}$.

Remark A.4.5 We have seen that $v \to \lim\limits_{\text{AA}} v$ is a linear operator, defined on a linear subspace of V and is norm-continuous there. Also it is clearly a *closed* operator. Thus in case V is a *Banach* space, its domain of definition (i.e. the set of vectors having AA-limit) is *closed*.

Proposition A.4.6 *Let B^* be the unit ball of V^* with the w^*-topology. Let $v \in V$. Then:*

a. *$\|v\|_A$ equals the maximum over G-invariant $K \subset B^*$ (or, if one wishes, over convex w^*-compact G-invariant $K \subset B^*$) of $\inf_{v^* \in K} \langle v, v^* \rangle$.*

b. *$\|v\|_{AA}$ equals the supremum over w^*-convex compact invariant $K \subset B^*$ of the minimum over G-invariant $K' \subset K$ (or, if one wishes, over minimal convex w^*-compact G-invariant $K' \subset K$) of $\sup_{v^* \in K'} \langle v, v^* \rangle$.*

Proof K, K' will refer to G-invariant convex w^*-compact subsets of B^*.

a. Follows from Thm. 3.5.4 b. for $p = \max$.

b. By a.,

$$\begin{aligned}
&\|v\|_{AA} = \sup_{w \in \mathcal{AV}(v)} \|w\|_A = \\
&= \sup_{w \in \mathcal{AV}(v)} \sup_K \min_{v^* \in K} \langle w, v^* \rangle = \\
&= \sup_K \sup_{w \in \mathcal{AV}(v)} \min_{v^* \in K} \langle w, v^* \rangle = \\
&= \sup_K \min_{K' \subset K} \max_{v^* \in K'} \langle v, v^* \rangle
\end{aligned}$$

The last equality following from Thm. 3.5.4 b. (for $p = \max$) applied to $-v$ instead of v.

QED

For the next theorem, we use the Ryll-Nardzewski Fixed-Point Theorem (see [Bo-E] Ch. IV App. for a proof):

Ryll-Nardzewski Fixed-Point Theorem: *Let V be a normed space, and K a convex non-empty weakly-compact subset of V. Let G be a group on affine norm-isometries of K. Then G has a fixed point in K*

Theorem A.4.7 *Let V be a G-normed space, i.e. a normed space on which a group G acts linearly isometrically. Suppose either V* is reflexive *or G* is amenable.

Then for every $v \in V$, $\|v\|_A = \|v\|_{AA}$ = the maximum of $|\langle v^*, v\rangle|$ over G-invariant *$v^* \in V^*$ with norm ≤ 1.*

Consequently,

$v \overset{AA}{\rightarrow} 0 \Leftrightarrow$

$\Leftrightarrow$ v has averages with arbitrarily small norm $\Leftrightarrow$

$\Leftrightarrow$ v is annulled by all G-invariant $v^* \in V^*$.

Proof The two assumptions: *V is reflexive* or *G is amenable*, have in common that they imply that in every bounded w^*-closed convex subset $K \subset V^*$ there is a G-fixed point. (for the case of V reflexive this follows from Ryll-Nardzewski's Thm.) Hence every minimal convex w^*-compact G-invariant subset of V^* is a singleton. Having said this, the assertion of the theorem follows from Prop. A.4.6 (note that the maximum of $|\langle v^*, v\rangle|$ over G-invariant v^*'s in the unit ball of V^* equals the maximum of $\langle v^*, v\rangle$).

QED

Let G act measure-preservingly on a probability space $(\Omega, \mathcal{B}, \mu)$.

The function spaces $L^p(\Omega)$, $1 \leq p \leq \infty$ are G-normed spaces. For $1 < p < \infty$ they are reflexive and Thm. A.4.7 applies. Hence every L^p-function with conditional expectation 0 w.r.t. the Boolean-σ-algebra of almost-invariant[10]subsets, in other words, any L^p-function annulled by the G-invariant members of the dual space, converges AA in L^p norm to 0.

G-Invariant members of L^p converge AA to themselves.

Now, any continuous functional $\phi \in L^{p'}$ annulled both by the functions with conditional expectation 0 and by the invariant functions must be 0, and the set of members of a G-Banach space with AA-limit is closed (Remark A.4.5). Hence in our case it is the whole space and by closeness we are allowed to say the same about L^1.

To conclude, we have the following "mean ergodic theorem", valid for *any group G* and not referring to Følner sequences.

Theorem A.4.8 *Let G act measure-preservingly on a probability space $(\Omega, \mathcal{B}, \mu)$. Let $1 \leq p < \infty$.*

Then every $f \in L^p$ converges in the AA (accumulating averages) sense in L^p-norm to an invariant $f_0 \in L^p$. f_0 is the conditional expectation of f w.r.t. the σ-algebra of almost-invariant[11]subsets. (in the L^2 case, f_0 is the orthogonal projection of f on the space of G-invariant functions). In other words, every (finite) average of f has (finite) averages arbitrarily close to f_0 in L^p norm.

QED

[10]see "Notations"
[11]see "Notations"

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